Lecture 5

Tuesday, September 29, 2020 12:05 PM

Lecture 5: Symmetries in Hamiltonian Dynamics

LAGRANGIAN DYNAMICS

$\gamma : [a,b] \to M^n$

$L : TM \to M$

Minimise $S(\gamma) = \int_\gamma L$.

$\rightsquigarrow$ (E-L eq^n) $\boxed{\frac{d}{dt}\partial_{\dot q} L = \partial_q L}$ n-eq^n. ODE, 2^{nd} order on TM

SYMMETRIES

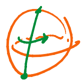
$\eta \in \Gamma(TM)$, $L : TM \to \mathbb{R}$

if $\mathcal{L}_\eta L = 0$ then $\langle \tilde\eta, -i_L^*(\lambda_{st})\rangle$ is a constant of motion

$dL(\tilde\eta)$

use these to integrate (solve ODE) w/o solving ode directly

Ex.



HAMILTONIAN DYNAMICS

$H : T^*M \to \mathbb{R}$

we consider X_H and its flow gives the dynamics:

Eq^n of motion "HAMILTON"

$$\begin{cases} \dot p_i(t) = \partial_{q_i} H \\ \dot q_i(t) = \partial_{p_i} H \end{cases}$$

$\rightsquigarrow$ $2n$ eq^n ODE, 1^{st} order on T^*M

SYMMETRIES

Today!

we'll define the Poisson bracket

(1) in coordinates (3) geometrically

(2) algebraic properties

$\{\,,\,\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$

$(f,g) \longmapsto \{f,g\}$.

§1. Poisson brackets:

what is a constant of motion in Hamiltonian dynamics?

Translating from a Lagr. system: η s.t. $\mathcal{L}_\eta L = 0 \rightsquigarrow I_\eta$ const motion

$$\frac{dI_\eta}{dt} = 0 \overset{\text{in } (q_i,p_i) \text{ coord}}{\Longleftrightarrow} \sum_{i=1}^n \partial_{q_i} I_\eta \cdot \dot q_i + \partial_{p_i} I_\eta \cdot \dot p_i = 0 \Longleftrightarrow$$

Hamilton's eq^n on a traj.

$$\Longleftrightarrow \underbrace{\sum_{i=1}^n \partial_{q_i} I_\eta\, \partial_{p_i} H - \partial_{p_i} I_\eta\, \partial_{q_i} H}_{} = 0.$$

take I_η, computable!

Def^n: Let $f,g \in C^\infty(T^*M)$, we define $\{f,g\} := \sum_{i=1}^n (\partial_{q_i} f)\cdot(\partial_{p_i} g) - (\partial_{p_i} f)(\partial_{q_i} g)$.

POISSON BRACKET

Lemma: The Poisson bracket satisfies:

(i) antisymmetric & bilinear (ii) Jacobi identity ("associativity")

(iii) Leibniz's rule: $\{f, gh\} = \{f,g\}\cdot h + g\cdot\{f,h\}$.

"$\{f,\cdot\}$ derivative"

algebra $C^\infty(T^*M)$ gets endowed with $\{\,,\,\}$.

SYMMETRIES $\Leftrightarrow \{I_\eta, H\} = 0$

point of $\{\cdot,\cdot\}$.

§2. Example of Kepler Potential:

particle of mass $m=1$ moving in $\mathbb{R}^3$, (q_1,q_2,q_3), with potential $V = V(r)$, $r = (q_1^2+q_2^2+q_3^2)^{1/2}$

(Ex: Planetary motion (1610s): $V(r) = 1/r$.)

Symmetries: energy H, coming from rotations (e.g. $SO(3)$).

time transl. Lie group of rotations

$$H(q,p) = \frac{1}{2}(p_1^2+p_2^2+p_3^2) + V(r)$$

(i) why is H a symmetry?

Bc $\{H,H\} = 0$ by antisym.

(ii) What are infinitesimal symmetries for rotations?

$$\begin{cases} \eta_1 = q_1\partial_{q_2} - q_2\partial_{q_1} \\ \eta_2 = q_2\partial_{q_3} - q_3\partial_{q_2} \\ \eta_3 = q_3\partial_{q_1} - q_1\partial_{q_3} \end{cases}$$

(Ex.) $\rightsquigarrow$

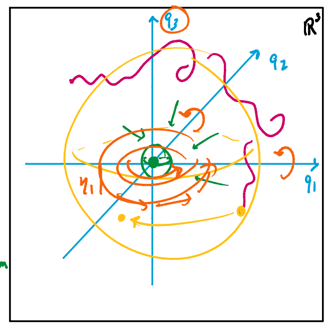
$$\boxed{\begin{aligned} I_{\eta_1} &= q_1p_2 - p_1q_2 \\ I_{\eta_2} &= q_2p_3 - p_2q_3 \\ I_{\eta_3} &= q_3p_1 - p_3q_1 \end{aligned}}$$

constant of motion

coord. of angular momentum

$\vec q \times \vec p = (I_{\eta_1}, I_{\eta_2}, I_{\eta_3}) \in \mathbb{R}^3$

$\vec q \in \mathbb{R}^3$, $\vec p \in \mathbb{R}^3$



Let's check I_{η_i}, $i = 1,2,3$ are symmetries:

$\boxed{\text{For } I_{\eta_1}:}$ Need to check $\{I_{\eta_1}, H\} = 0$. $I_{\eta_1} = q_1p_2 - q_2p_1$, $H = \frac{1}{2}p^2 - V(r)$.

$$\{I_{\eta_1}, H\} = \left[\left(\partial_{q_1} I_{\eta_1}\cdot\partial_{p_1} H - \partial_{p_1} I_{\eta_1}\,\partial_{q_1} H\right)\right] + \left[(\partial_{q_2} I_{\eta_1})\cdot(\partial_{p_2} H) - (\partial_{p_2} I_{\eta_1})\cdot(\partial_{q_2} H)\right]$$

$$= \left(\cancel{p_2}\cdot p_1 + q_2\cdot \cancel{V'(r)}\cdot\frac{q_1}{r}\right) + \left(-\cancel{p_1}\cdot p_2 - q_1\cdot\cancel{V'(r)}\,\frac{q_2}{r}\right) = 0$$

Note $\{I_{\eta_1}, I_{\eta_2}\} \neq 0$ (check!) so they are not commuting symmetries.

But I_{η_3} and $I_{\eta_1}^2 + I_{\eta_2}^2 + I_{\eta_3}^2$ are, i.e. $\{I_{\eta_3}, I_{\eta_1}^2 + I_{\eta_2}^2 + I_{\eta_3}^2\} = 0$. ▨

try (use) Leibniz

$\{I_{\eta_3}, I_{\eta_1}^2\} = 2\cdot I_{\eta_1}\{I_{\eta_3}, I_{\eta_1}\}$

(Check: $\{I_{\eta_1}, I_{\eta_2}\} = I_{\eta_3}$, $\{I_{\eta_2}, I_{\eta_3}\} = I_{\eta_1}$, $\{I_{\eta_3}, I_{\eta_1}\} = I_{\eta_2}$) $\leftarrow$ $so(3)$ Lie algebra