Lecture 6

Tuesday, September 29, 2020 12:05 PM

Lecture 6 : Poisson Dynamics

(we're in Lect. 3 in textbook)

Q configuration space

T^*Q phase space

(q, p) positions, momenta

$\gamma : [0,1] \to Q$

$(C^\infty(T^*Q), \{\cdot,\cdot\})$ Poisson algebra

$\overset{\shortparallel}{G(T^*Q)}$

Choose $H \in G(T^*Q)$ Hamiltonian :

(i) Hamilton's eq^n: $f \in G(T^*Q)$ is by def. an observable.

general observable f

$$\frac{d}{dt} f = \{f, H\} \rightsquigarrow \begin{cases} \dot{q} = \{q, H\} & (f = q) \\ \dot{p} = \{p, H\} & (f = p) \end{cases}$$

(ii) $f \in G(T^*M)$ constant of motion iff

$$\{f, H\} = 0.$$

Rmk: Thus far $\{\cdot,\cdot\}$ defined only in coordinates. $\{f,g\} = \sum_{i=1}^{n} (\partial_{q_i} f \cdot \partial_{p_i} g - \partial_{q_i} g \, \partial_{p_i} f)$

The Poisson bracket in T^*Q

: given $f \in G(T^*Q)$ we had X_f v.f. on T^*Q

$$\left.\begin{aligned} f, g \in G(T^*Q) &\rightsquigarrow \{f, g\} \text{ fct.} \\ \wr \qquad &\qquad \shortparallel \\ X_f, X_g &\rightsquigarrow \omega(X_f, X_g) \text{ fct.} \end{aligned}\right]$$

in general, we can define $\{f, g\} := \omega(X_f, X_g)$.

Ex: $Q = \mathbb{R}_q$, T^*Q coord. (q,p), $\omega = dq\,dp$.

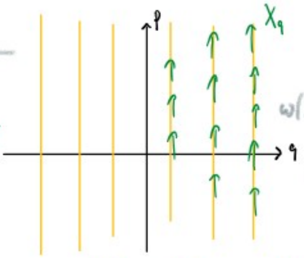
$f = q \rightsquigarrow X_q = \partial_p$, $f = q^2 \rightsquigarrow X_{q^2} = p \cdot \partial_p$.

$f = p \rightsquigarrow X_p = -\partial_q$, $\omega(X_f, X_f) = -df(X_f)$

$\overset{\shortparallel}{0}$

(recall X_f defined by $\omega(X_f, -) = -df$)

$T(\text{level set}) = \ker(df)$



$\{q,p\} = \boxed{1}$ ✓

$\shortparallel$

$\omega(X_q, X_p)$

$\shortparallel$

$dq\,dp(\partial_p, -\partial_q)$

$\shortparallel$

$\boxed{1}$

$\alpha_1 \wedge \alpha_2 (X) := \alpha_1(X) \cdot \alpha_2 - \alpha_2(X) \cdot \alpha_1$

2-form

§ 2. Poisson dynamics

⚠ no longer asking that $\underset{\uparrow}{M} = T^*Q$, e.g. $M = \mathbb{R}^3$.

space we study

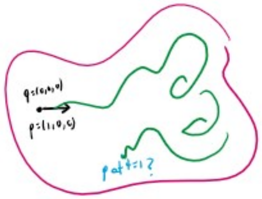
Many manifolds have $(G(M), \{\cdot,\cdot\})$ Poisson algebra. Thus, choosing $H \in G(M)$ allows us to consider dynamics. Given $f \in G(M)$ observable, its evolution is

$$\frac{d}{dt} f = \{f, H\} \longleftarrow$$

this is to hold along a trajectory $\gamma : [a,b] \to M$

Geometrically, $f \longmapsto \{f, H\}$ is a derivation (i.e. satisfies Leibniz)

this gives a vector field "X_H" in M.



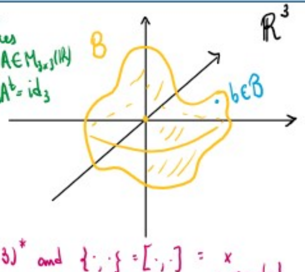
Ex: For $M = T^*Q$, then X_H is equivalent to $f \mapsto \{f, H\}$. $\overset{\curvearrowright}{\text{fct.}}$

X v.f. $X(f) := \mathcal{L}_X f := df(X)$ fct. This exercise says $X_H(f) = \{f, H\}$.

§ 3. An example : Rigid body

(i) Since $\text{dist}(b, 0)$ stays constant, the evolution of b is given by $w(t) = Q(t) \cdot b$, where $Q(t) \in SO(3)$

rotation matrices i.e. $A \in M_{3\times3}(\mathbb{R})$ s.t. $AA^t = id_3$



(ii) The velocity $\dot{w}(t) = \dot{Q}(t) \cdot b = \dot{Q} \cdot (Q^{-1} w)$

in $T_{Q(t)} SO(3) \cong so(3)$

$\hookrightarrow M = so(3)^*$ and $\{\cdot,\cdot\} = [\cdot,\cdot] = \underset{\text{cross product}}{\times}$

$Id \in SO(3)$,

this is the Lie algebra of $SO(3)$, denoted $so(3)$.

$(\underset{\uparrow}{I} + \varepsilon A)$ is in $SO(3)$ $\Leftrightarrow (Id + \varepsilon A)(Id + \varepsilon A)^t = id \Leftrightarrow Id + \varepsilon(\underbrace{A + A^t}_{\overset{\shortparallel}{0}}) + O(\varepsilon^2)$.

varying in $SO(3)$

$\Rightarrow T_{id} SO(3) = \{A \in M_{3\times3} \text{ s.t. } A + A^t = 0\}$ ↙ skew matrices, 3 $\dim_{\mathbb{R}}$ v. space $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$