Lecture 7

Tuesday, September 29, 2020 12:05 PM

Lecture 7: Rigid body dynamics (and Lie algebras)

$G = SO(3)$, $\mathfrak{g} := T_e G =: so(3)$

$A \in M_3(\mathbb{R}) : AA^t = id$ $\quad$ $A \in M_3(\mathbb{R}) : A + A^t = 0$

For Poisson brackets:

(1) $(\mathbb{R}^3, \times)$ is Poisson, $\quad$ (2) $(\mathfrak{g} = so(3), [\,,\,])$

cross product

matrix commutator: $[A,B] := AB - BA$

Prop: $(\mathbb{R}^3, \times) \overset{\text{Poisson}}{\cong} (so(3), [\,,\,])$.

$A \in so(3)$, $A = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$

The isomorphism is given by

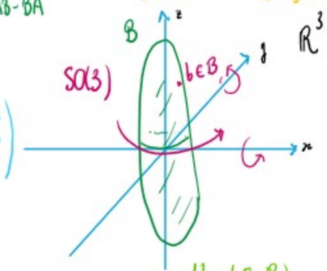
$$(x_1, x_2, x_3) \in \mathbb{R}^3 \xrightarrow{\varphi} A(\vec{x}) = \begin{pmatrix} 0 & -x_3 & x_1 \\ x_3 & 0 & -x_2 \\ -x_1 & x_2 & 0 \end{pmatrix}. \square$$

We are saying $\varphi(v \times w) = [\varphi(v), \varphi(w)]$

Rmk: $A(v) \cdot w = v \times w$.

$\leadsto$ in general, $\mathfrak{g}^*$ always has a canonical Poisson bracket (from $[\,,\,]$ Lie bracket)

$\mathrm{Hom}(\mathfrak{g}, \mathbb{R})$



§1. Rigid body setup: $b \in B$, $w(t) = Q(t) \cdot b$, $Q(t) \in SO(3)$.

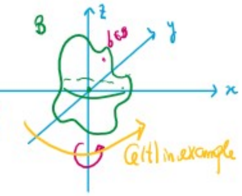
To understand the kinetic term in Hamiltonian we need $\dot{w}(t)$ and dual momentum.

$$\dot{w}(t) = \dot{Q}(t) \cdot b = \boxed{\dot{Q}(t) \cdot Q^{-1}(t)} \cdot w(t).$$

$\boxed{\dot{Q}(t) \cdot Q^{-1}(t)} \in SO(3)$

rotation along (x_1, x_2, x_3) given $A(x)$

$(0\,0\,1) \leadsto A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$



Ex: Rotating B along z-axis gives

$$Q(t) = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q'(t) = \begin{pmatrix} -\sin(t) & \cos(t) & 0 \\ -\cos(t) & -\sin(t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \in T_{Q(t)} SO(3)$$

$T_{id} SO(3) \ni$

$$\dot{Q}(t) \cdot Q^{-1}(t) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in so(3), \text{ i.e. antisymmetric.}$$

$$Q^{-1}(t) = Q(-t) = \begin{pmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\square$

Kinetic energy & Hamiltonian

in a rigid body $\exists!\, x$ s.t. $\dot{w} = x \times w$ (x: axis of rot.)

$$(1)\quad K_b = \frac{m_b}{2} \|\dot{w}(t)\|^2 = \frac{m_b}{2} \| x(t) \times w(t) \|^2 = \frac{m_b}{2} \| Q^{-1}(t) \cdot (x \times w(t)) \|^2 = \overline{X}^t \cdot (\Pi(b)) \cdot \overline{X}$$

matrix 3x3 : inertia product

$\hookrightarrow \Pi = \Pi(b)$, $b \in B$

K_b: Kinetic at b

This is at a point $b \in B$: $K = \int_B K_b\, d\mu_b = \overline{X}^t \cdot \Pi \cdot \overline{X}$

K: kinetic energy of B

$= \frac{m_b}{2} \|\overline{X} \times b\|^2$, $\quad \overline{X} := \underbrace{Q^{-1}}_{\in so(3)} x \in \mathbb{R}^3$

The dual momentum is given by $M = \Pi \cdot \overline{X} = (M_1, M_2, M_3) \in so(3)^*$ $\Big\} \Rightarrow \boxed{H : so(3)^* \longrightarrow \mathbb{R}}$, $H(M_1, M_2, M_3) = \frac{1}{2} M^t \cdot \Pi^{-1} \cdot M$.

Rmk: we'll assume $\Pi = \mathrm{diag}(I_1, I_2, I_3)$ (moments of inertia). So

$$\boxed{H(M_1, M_2, M_3) = \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3} \right)}. \square$$

$$\Rightarrow \boxed{\text{Eq}^n \text{ of motion} \begin{cases} \dot{M}_1 = \{M_1, H\} \\ \dot{M}_2 = \{M_2, H\} \\ \dot{M}_3 = \{M_3, H\} \end{cases}}$$

$\leftsquigarrow$ e.g. $\{M_1, H\} = (M_1, M_2, M_3) \cdot \left((1,0,0) \times \left(\frac{M_1}{I_1}, \frac{M_2}{I_2}, \frac{M_3}{I_3} \right) \right)$

$so(3)^*$ same as $\times$

$$= (M_1, M_2, M_3) \cdot \left(0, -\frac{M_3}{I_3}, \frac{M_2}{I_2} \right) = \frac{M_2 M_3}{I_2 I_3} (I_3 - I_2)$$

§2. Symmetries: $M = \mathfrak{g}^* = so(3)^*$, $H(M) = \frac{1}{2} M^t \cdot \Pi^{-1} \cdot M$,

(i) The energy is a conserved quantity, since the level sets of H are ELLIPSOIDS:

$$H = \frac{1}{2}\left(M_1^2/I_1 + M_2^2/I_2 + M_3^2/I_3 \right)$$

(ii) A 2^{nd} symmetry exists: $J = M_1^2 + M_2^2 + M_3^2$.

Proof: we want $\{J, H\} = 0$.

$\{M_1, H\} = \frac{M_2 M_3}{I_2 I_3}(I_3 - I_2)$

First, $\{M_1^2, H\} = 2 \cdot \{M_1, H\} \cdot M_1 = 2 M_1 M_2 M_3 \cdot (I_3 - I_2) / I_2 I_3$

$\{M_2^2, H\} = 2 M_1 M_2 M_3 \cdot \left(\frac{I_1 - I_3}{I_1 I_3} \right)$

$\{M_3^2, H\} = 2 M_1 M_2 M_3 \cdot \left(\frac{I_2 - I_1}{I_1 I_2} \right)$

$$\{J, H\} = 2 \cdot M_1 M_2 M_3 \left(\frac{I_1(I_3 - I_2) + \ldots + \ldots}{I_1 I_2 I_3} \right) = 0. \quad \square$$