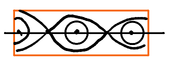
Lecture 8

Tuesday, September 29, 2020 12:05 PM

Lecture 8 : Integrable Systems & Beyond : a quick tour to 3 results by V.I. Arnol'd and its collaborators → see upcoming PSet 3 for more

Dynamical : Hamiltonian $H \in C^\infty(M)$ and Poisson bracket on $C^\infty(M)$ $\rightsquigarrow$ $\dot{f} = \{f, H\}$.

Examples : (1) Harmonic oscillator : $M = T^*\mathbb{R}$, $H(q,p) = \frac{p^2}{2m} + \omega q^2$.



$\{f,g\}$ is the same $\omega_{st}(X_f, X_g)$.

(2) Kepler's Problem : $M = T^*\mathbb{R}^3$, $H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(r)$ ↪ radius $\rightsquigarrow$ $SO(3)$ symmetry

(3) Rigid body : $M = SO(3)^* = \mathbb{R}^3$ (not symplectic) with Poisson bracket $\cong \times$ cross prod. ⟵ see PSet 3 for more example : $C^\infty(\mathfrak{g}^*)$ is Poisson.

Today : Integrable systems ; i.e. (M^{2n}, ω) symp. phase space with $f_1, \dots, f_n \in C^\infty(M)$ s.t.

(1) (Involution) $\{f_i, f_j\} = 0$, $\forall i,j \in [1,n]$

(2) assume that $(f_1, \dots, f_n) : M^{2n} \longrightarrow \mathbb{R}^n$ is non-singular, i.e.

$$df_1 \wedge \dots \wedge df_n \neq 0$$

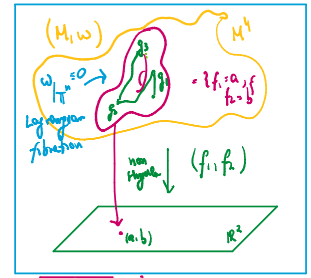
↗ e.g. look at Toda system (Toda lattice)

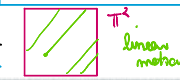
Thm. (Arnol'd - Liouville) Let (M^{2n}, ω) and $(f_1, \dots, f_n)$ be an integrable system. Choose $c \in \mathbb{R}^n$ such that $f^{-1}(c)$ is compact$^{(*)}$, where $f = (f_1, \dots, f_n) : M \longrightarrow \mathbb{R}^n$.

(*) if not $\mathbb{T}^k \times \mathbb{R}^{n-k}$

Then (i) $f^{-1}(c) \overset{\text{diff.}}{\cong} \mathbb{T}^n$, the n-torus $\mathbb{T}^n := S^1 \times S^1 \times \overset{(n)}{\dots} \times S^1$

(ii) The dynamics in $\mathbb{T}^n$ are given linearly in certain explicit coord. (ACTION-ANGLE)





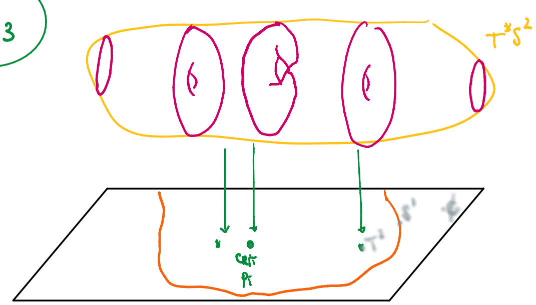
Proof-Sketch of (i) : we have $f_1 \dots f_n$, since (M, ω) is symplectic, we get v.f. $X_{f_1}, \dots, X_{f_n}$.
Since $\{f_i, f_j\} = 0$ and $\omega(X_{f_i}, X_{f_j}) = 0$ and $[X_{f_i}, X_{f_j}] = 0$. Hence the flows $g^t_{f_i}, g^t_{f_j}$ commute.
($n=2$ (3, 2.1))
That means that $\exists g : \mathbb{R}^n \longrightarrow f^{-1}_w(c)$ given by flow. If $f^{-1}(c)$ compact, $f^{-1}(c) \cong \mathbb{R}^n / \text{discrete sg.} \cong \mathbb{R}^n / \mathbb{Z}^n =: \mathbb{T}^n$ ∎

Example : (Spherical Pendulum) $(M = T^*S^2, \omega_{st})$, $H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \underbrace{V}_{q_3}$

The claim is that J = z - coord. of angular momentum :

$$T^*S^2 \xrightarrow{(H,J)} \mathbb{R}^2$$ is an integrable system.

See PSet 3



If perturb H, the system is no longer integrable
BUT the main result of KAM thy is that some tori do persist !