Lecture 9

Tuesday, September 29, 2020 12:05 PM

Lecture 9 : Probability in Mechanics

← useful for quantum mechanics, and also statistical mechanics (e.g. thermodyn)

CLASSICAL MECHANICS

(1) Lagrangian mech. : minimize action $S = \int L \rightsquigarrow$ E-L eq^{ns}

(2) Hamiltonian mech. : Hamiltonian $H \in C^\infty(M)$

then dynamics are flow of $X_H \rightsquigarrow$ Hamilt.'s eq^{ns}

(3) Poisson dynamics : $f \in G(M)$ observable and $(G(M), \{\cdot,\cdot\})$ Poisson $\rightsquigarrow \dot{f} = \{f, H\}$

REALITY NEED - - - →

"$\hbar \to 0$"

← high freq. limit

QUANTUM MECHANICS

(1) Heisenberg quantization : matrix mechanics

(2) Schrödinger quantization : wave mechanics & PDEs

(3) Weyl quantization : deformation of $\{\cdot,\cdot\}$

(4) Feynman quantization : integral rep^n & path integral

(5) Geometric & Bargmann quant. : L^2-hol. sections

(6) SUSY quant. (7) BRST quant. (8) More!

generalize particles $\gamma : [0,1] \to M$ } to fields, i.e. sections of fibre bundles : e.g. connections

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CLASSICAL FIELD THEORY $\rightsquigarrow$ Klein-Gordon, Yang-Mills, etc. - - "no-go" thms require - - → QUANTUM FIELD THEORY

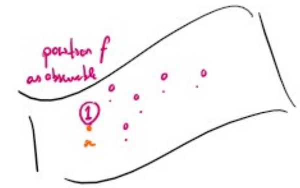
§ 1. Observables and States

: $G(M)$ algebra observables (classically $C^\infty(M)$)

Def: A state for a system with observables $G(M)$ is

$$\mu : G(M) \longrightarrow \mu^{\mathbb{R}} \quad \leftarrow \text{set of prob. distr. on } \mathbb{R}$$
$$f \longmapsto \mu_f.$$

The set of states is $S(M)$. (+ additional properties)



Ex: (1) Classically : $\mu = \mu_\delta$, i.e. $f \longmapsto \mu_{\delta(f(x))}$ for a point x.

(2) $\mu(E) := \dfrac{\mu(E \cap C)}{\mu(C)}$

Rmk: in quantum mechanics the uncertainty comes from nature (as of 2020) and not from failing of measurement

§ 2. Some comments on states

START w/ μ a state

: given $\mu_f \in \mu^{\mathbb{R}}$, we can measure $\mu_f(\lambda) := \mu_f((-\infty, \lambda])$. $\mu_{\phi \cdot f} = \mu_f(\phi^{-1})$

→ $\mathbb{E}_\mu(u_0(\lambda - f))$

1. Note that knowing $\mu_f(\lambda)$ recovers μ_f, because $\mu_f([a,b]) := \mu_f(b) - \mu_f(a)$.

2. Also we consider the expectation :

$$\mathbb{E}_\mu(f) := \langle f \mid \mu \rangle := \int_{\mathbb{R}} x \cdot d\mu_f \left(= \int x \cdot \rho \, dx\right), \rightsquigarrow$$

know for all f recovers $\mu_f(\lambda)$ using Heaviside step $\rightsquigarrow$ state is μ_f know

and the deviation : $\sigma_\mu(f) := \sqrt{\mathbb{E}_\mu(f^2) - \mathbb{E}_\mu(f)^2}$. $\rightsquigarrow$ featuring in Heisenberg uncertainty ppl.

3. For a general symplectic manifold (M^{2n}, ω) :

(1) we need a volume : $\Omega := \omega^n$.

(2) $\mu : G(M) \longrightarrow \mu^{\mathbb{R}}$, we assume it is linear and furthermore ask $\int_M \rho = 1$ and $\rho \geq 0$.

$\mathbb{E}_\mu(f+g) = \mathbb{E}_\mu(f) + \mathbb{E}_\mu(g)$

$\mu : G(M) \longrightarrow \mu^{\mathbb{R}}$ is a state, which is determined by $\mathbb{E}_\mu(\cdot)$.

We want it to be of the form :

$$\mathbb{E}_\mu(f) := \int_M f \cdot \overbrace{\rho_\mu \cdot \underbrace{\Omega}_{\text{volume form}}}^{\text{measure on manifold}} \in \mathbb{R} \rightsquigarrow \rho$$

"a state"

§ 2. States vs. Observables

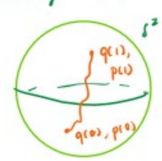
. μ state $\rightsquigarrow \rho$ function, $\underline{H \text{ hamiltonian}} \rightsquigarrow X_H \overset{\text{flow}}{\rightsquigarrow} \phi_H^t$

$\gamma : [0,1] \longrightarrow M$

HEISENBERG'S PICTURE : observables $f \in G(M)$

evolve in time via $\dot{f}_t = \{f, H\}$.

$\{\cdot, H\}$ RHS is v.f. on M so is LHS



In contrast, we ask

$\dot{\rho}_t = 0$, i.e. state are constant.

SCHRÖDINGER'S PICTURE : observables $f \in G(M)$

stay the same $\dot{f}_t = 0$, but in contrast

$$\dot{\rho}_t = -\{\rho_t, H\}.$$

$\rho_t := \rho(\phi_H^{-t})$, where $\phi_H^t \in \mathrm{Diff}(M)$.

Prop: (Equivalency for these 2 perspectives) $\mathbb{E}_\mu(f_t) = \mathbb{E}_{\mu_t}(f)$, here μ_t associated ρ_t.