MAT 265: PROBLEM SET 1

DUE TO FRIDAY OCT 9 AT 9:00AM

Abstract. This is the first problem set for the graduate course Mathematical Quantum Mechanics in the Fall Quarter 2020. It was posted online on Wednesday Sep 30 and is due Friday Oct 9 at 9:00am via online submission.

Purpose: The goal of this assignment is to review and practice the basic concepts from Mathematical Quantum Mechanics (MAT265). In particular, we would like to become familiar with many examples of topological spaces, including fiber bundles and covering spaces, as well as with the algebra appearing from homotopy groups..

Task and Grade: Solve three of the six problems below. Each Problem is worth $33.\hat{3}$ points.The maximum possible grade is 100 points. Despite the task being three problems, I strongly encourage you to work on the six problems.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

Grade:

Textbook: We will use "A Brief Introduction to Physics for Mathematicians" by I. Dolgachev. Please contact me *immediately* if you have not been able to get a copy of any edition.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. (The Length Functional) Consider the space of smooth functions

$$\Omega(p_0, p_1) = \{f \in C^\infty([0,1]) : f(0) = 0, f(1) = 1\},$$

whose parametrized smooth graphs

$$\mathrm{gr}(f) := \{(x_1, x_2) \in \mathbb{R}_2 : (x_1, x_2) = (t, f(t)), t \in [0,1]\} \subseteq \mathbb{R}^2$$

in the plane $\mathbb{R}^2$ start at $p_0 = (0,0)$ and end at $p_1 = (1,1)$.

(a) Show that the length of the curve $\mathrm{gr}(f)$ is given by the functional

$$\lambda : \Omega(p_0, p_1) \longrightarrow \mathbb{R}, \quad \lambda(f) = \int_0^1 \sqrt{1 + (f'(t))^2}dt.$$

(b) Show that $\lambda : \Omega(p_0, p_1) \longrightarrow \mathbb{R}$ is unbounded above, bounded below and attains its minimum value.

(c) Find all the critical points of λ. In particular, prove that $f \in \Omega(p_0, p_1)$ is a minimum if and only if $\mathrm{gr}(f)$ is a straight segment.

Problem 2. (Two Lessons) Consider the functional

$$\Sigma_1 : \Omega(p_0, p_1) \longrightarrow \mathbb{R}, \quad \Sigma_1(f) = \int_0^1 \sqrt{f(t)^2 + (f'(t))^2}dt.$$

(a) Show that the infimum of Σ_1 is one, i.e. $\inf(\Sigma_1) = 1$.

(b) Show that the minimum of Σ_1 does not exist.

Thus, first lesson: *positive* functionals may not necessarily have a minimum.

Consider the functional

$$\Sigma_2 : \Omega(p_0, p_1) \longrightarrow \mathbb{R}, \quad \Sigma_2(f) = \int_0^1 (f'(t))^2 + f(t)(\sin(t) + t^2)dt.$$

(a) Show that Σ_2 is *convex*.

(b) Show that a minimum of Σ_2 exists, and conclude that it is unique.

Thus, second lesson: *convex* functionals have *unique* minimum.

In general, some of the basic questions to ask yourself when a functional is given are whether it is positive and/or convex, whether it admits a minimum and whether it has any other critical points.

Problem 3. (Potentials) Let $M \in \mathbb{R}^+$. For each of the following potentials $V : \mathbb{R}^n \longrightarrow \mathbb{R}$, consider the dynamical system with action

$$S(\gamma) := \int_0^M \left(\frac{1}{2} \|\gamma'(t)\|^2 - V(\gamma(t)) \right) dt,$$

where the position $\gamma(t)$ of the particle at time $t \in [0, M]$ is given by the smooth function $\gamma : [0, M] \longrightarrow \mathbb{R}^n$. Cartesian coordinates in $\mathbb{R}^n$ are $(q_1, \ldots, q_n)$ and $q = q_1$ for $n = 1$.

(a) Consider $n = 1$ and $V(q) = q^4 - q^2 + 1$. Assume that the particle is initially at rest at $q = \gamma(0) = 0.70719$. Show that, independent of M, the particle will stay within the interval $(0.2, 0.9)$, and thus never get near the origin $q = 0$.

(b) Consider $n = 1$ and $V(q) = q^4 - q^2 + 1$. Assume that the particle is initially thrown from $q = \gamma(0) = 2$ with speed $\gamma'(0) = 9$. Show that the particle will pass through $q = 0$ infinitely many times if $M \to \infty$. Would this still be true if the particle was initially at rest, i.e. $\gamma'(0) = 0$?

(c) Consider $n = 1$ and $V(q) = 4q^5 - 8q^4 - 5q^3 + 10q^2 + q - 2$. Assume that the particle is initially at $q = \gamma(0) = -0.80041$ and thrown to the right with velocity $\gamma'(0) = 23$. Describe the *long-term* behavior of the particle, i.e. the position of $\gamma(t)$ for $t \gg 1$ and M large enough.

(d) Consider $n = 1$ and $V(q_1, q_2) = -7q_1^2 - 19q_2^2$. Show that a particle located at $\gamma(0) = (0, 0)$ which is slightly pushed in *any direction*, i.e. $\gamma'(0) = \varepsilon \cdot v$ for any non-zero $v \in \mathbb{R}^2$, will move farther and farther away from the origin $(0, 0)$.

Problem 4. (The fastest way of sliding between two points) Consider the space

$$\Omega^-(p_0, p_1) := \{f \in C^\infty([0, 1]) : f(0) = 1, f(1) = 0, f \text{ decreasing}\},$$

whose graphs $\text{gr}(f)$ we interpret as possible paths followed by a particle being dropped from $p_0 = (0, 1)$ that ends at $p_1 = (1, 0)$. The only force acting on the particle at point $(t, f(t))$ is given by the gravitational potential energy $mgf(t)$, if the particle has mass m. This problem is about finding the *fastest* way a particle in these circumstances can slide down from $(1, 0)$ to $(0, 1)$ following a curve $\text{gr}(f)$, $f \in \Omega^-(p_0, p_1)$.[1]

(a) Find the functional $\mathcal{B} : \Omega^-(p_0, p_1) \longrightarrow \mathbb{R}$ such that $\mathcal{B}(f)$ gives the time that the particle takes to move from p_0 to p_1.

(b) Show that the straight line, given as the graph of the function $l(t) = 1 - t$, is *not* a minimum for $\mathcal{B}$, i.e. there exists $g \in \Omega^-(p_0, p_1)$ such that $\mathcal{B}(g) < \mathcal{B}(l)$.[2]

(c) Find the Euler-Lagrangian equation associated to $\mathcal{B}$, i.e. find the equation for the critical points of $\mathcal{B}$.

(d) Suppose $m = 1$, find a minimum for $\mathcal{B}$, i.e. find the *fastest* (not the *shortest* !) way to slide down from $(1, 0)$ to $(0, 1)$ along a curve $\text{gr}(f)$.

[1] Equivalently, if you ever visited a water park: design the fastest water slide.

[2] In particular, straight water slides are terrible.

Problem 5. (Just kinetic energy) Let $a, b \in \mathbb{R}$, $a \leq b$. In this problem we study particles moving in a space M only subject to their kinetic energy, with *no* potential energy (and in particular, no force being exerted).

(a) Consider $M = \mathbb{R}^2$ with its standard metric $g = dq_1 \otimes dq_1 + dq_2 \otimes dq_2$, i.e. for tangent vectors $v = (v_1, v_2), w = (w_1, w_2) \in T_{(q_1,q_2)}\mathbb{R}^2$, the inner product is $\langle v, w\rangle = v_1w_1 + v_2w_2$. Write the Euler-Lagrange equations associated to the functional

$$S(\gamma) := \int_a^b \|\gamma'(t)\|^2 dt, \quad \gamma : [a, b] \longrightarrow \mathbb{R}^2.$$

(b) Find all the critical points for the action functional $S(\gamma)$ in Part (a).

(c) Consider two particle trajectories γ_1, γ_2 initially starting at $\gamma_1(0) = \gamma_2(0) = (0, 0) \in \mathbb{R}^2$. One is thrown to the right with velocity $\gamma_1'(0) = (1, 0)$, and the other is thrown to right *and slightly* up, with velocity $\gamma_2'(0) = (1, \varepsilon)$. Study how the distance between $\gamma_1(t)$ and $\gamma_2(t)$ depends on ε for $t \gg 1$: is it a polynomial dependency, a trigonometric one, an exponential ?

(d) Consider the hyperbolic plane $M = \mathbb{H} = \{(q_1, q_2) \in \mathbb{R}^2, q_2 > 0\}$ with its (constant negative curvature) metric $g = \frac{1}{q_2^2}(dq_1 \otimes dq_1 + dq_2 \otimes dq_2)$. Thus, the inner product of tangent vectors $v, w \in T_{(q_1,q_2)}\mathbb{R}^2$ is $\langle v, w\rangle = \frac{1}{q_2^2}(v_1w_1 + v_2w_2)$. Write the Euler-Lagrange equations associated to the functional

$$S(\gamma) := \int_a^b \|\gamma'(t)\|^2 dt, \quad \gamma : [a, b] \longrightarrow \mathbb{R}^2.$$

(e) Show that the "straight" trajectory $\gamma(t) = (t, 1)$ is *not* a critical points of the action $S(\gamma)$, and thus no particle follows this trajectory.

(f) Find the trajectory $\gamma : [a, b] \longrightarrow \mathbb{H}$ of a particle which starts to $(0, 1) \in \mathbb{H}$ and is thrown to the right with velocity $\gamma'(0) = (1, 0)$.

(g) Find the trajectory $\gamma : [a, b] \longrightarrow \mathbb{H}$ of a particle which starts to $(0, 1) \in \mathbb{H}$ and is thrown upwards with velocity $\gamma'(0) = (0, 1)$.

(h) Consider two particle trajectories γ_1, γ_2 initially starting at $\gamma_1(0) = \gamma_2(0) = (1, 0) \in \mathbb{H}$. One is thrown to the right with velocity $\gamma_1'(0) = (1, 0)$, and the other is thrown to right *and slightly* up, with velocity $\gamma_2'(0) = (1, \varepsilon)$. Study how the distance between $\gamma_1(t)$ and $\gamma_2(t)$ depends on ε for $t \gg 1$: is it a polynomial dependency, a trigonometric one, an exponential ?

(i) Consider the 2-sphere $M = \mathbb{S}^2 = \{(q_1, q_2, q_3) \in \mathbb{R}^3 : q_1^2 + q_2^2 + q_3^2 = 1\}$ with its standard round metric, given by the restriction of $g = dq_1 \otimes dq_1 + dq_2 \otimes dq_2 + dq_3 \otimes dq_3$ to $\mathbb{S}^2$. Consider two particle trajectories γ_1, γ_2 initially starting at $\gamma_1(0) = \gamma_2(0) = (0, 0, 1) \in \mathbb{H}$. One is thrown to the right with velocity vector $\gamma_1'(0) = (1, 0, 0)$, and the other is thrown to right *and slightly* up, with velocity $\gamma_2'(0) = (1, \varepsilon, 0)$. Study how the distance between $\gamma_1(t)$ and $\gamma_2(t)$ depends on ε for $t \gg 1$: is it a polynomial, trigonometric or exponential ?

Problem 6. (The Harmonic Oscillator) An harmonic oscillator is an oscillator that is neither driven nor damped, i.e. a particle at position $x(t) \in \mathbb{R}$ in time $t \in [0, \infty)$, that experiences no friction and a single force F which pulls the mass in the direction of the point $0 \in \mathbb{R}$ and depends only on the position $x(t)$ of the mass. A mass-spring system or a pendulum with small oscillations are real-life examples close to harmonic oscillators. The potential energy experienced at the position $x \in \mathbb{R}$ is $V(x) = \frac{1}{2}x^2$, and thus the system has action

$$S(\gamma) := \int_0^1 \left(\frac{m}{2} \|\gamma'(t)\|^2 - \frac{1}{2}\gamma(t)^2 \right) dt, \quad \gamma : [0, \infty) \longrightarrow \mathbb{R},$$

where $\gamma(t) \in \mathbb{R}$ is the position of the particle at time $t \in [0, 1]$.[3]

(a) Suppose the particle is at rest in the position $0 \in \mathbb{R}$ at time $t = 0$. Argue, without taking any derivatives, that the particle will stay at $0 \in \mathbb{R}$ for the entire period $t \in [0, \infty)$.

(b) Find the Euler-Lagrange equations for the critical points $\gamma \in C^\infty([0, \infty))$ of the action $S(\gamma)$. Conclude that the motion of the particle is *periodic* and find its period T, i.e. $\gamma(t) = \gamma(t + T)$ for some constant $T = T(m)$.

(c) Show that the Hamiltonian

$$H : T^*\mathbb{R} \longrightarrow \mathbb{R}, \quad H(q, p) := \frac{1}{2m} \|p\|^2 + \frac{1}{2}q^2$$

leads to the same dynamical system, i.e. equations of motion, than the action $S(\gamma)$ above. Here $q \in \mathbb{R}$ is a coordinate on the base $\mathbb{R}$ of the cotangent bundle $T^*\mathbb{R}$, and $p \in \mathbb{R}$ is a coordinate on the $\mathbb{R}$-fibers of the projection $T^*\mathbb{R} \longrightarrow \mathbb{R}$.

(d) Draw the level sets of the Hamiltonian $H : T^*\mathbb{R} \longrightarrow \mathbb{R}$ and interpret them physically, in terms of the position and velocity (or momentum) of the particle. Describe all energies $E \in \mathbb{R}$ for which there is a state with energy E.

(e) Let $a \in \mathbb{R}^+$ be a real number. Suppose a particle of mass $m = 1$ is at $\gamma(0) = a \in \mathbb{R}^+$ at $t = 0$ and it is thrown towards $x = 0$ with initial velocity $\gamma'(0) = -a$. Study how the time it takes for it to be back at $x = a$ depends on the choice of $a \in \mathbb{R}^+$?

[3] Oftentimes $\gamma(t)$ is denoted $x(t)$.