MAT 280: PROBLEM SET 1

DUE TO FRIDAY OCT 1 AT 9:00PM

Abstract. This is the first problem set for the graduate course Contact and Symplectic Topology in the Fall Quarter 2021. It was posted online on Thursday Sep 23 and is due Friday Oct 1 at 9:00pm via online submission.

Purpose: The goal of this assignment is to review and practice the basic concepts from Mathematical Quantum Mechanics (MAT265). In particular, we would like to become familiar with many examples of topological spaces, including fiber bundles and covering spaces, as well as with the algebra appearing from homotopy groups..

Task and Grade: Solve one of the five problems below. The maximum possible grade is 100 points. Despite the task being one problem, I strongly encourage you to work on the five problems.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct.

Problem 1. Let $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ be considered as a smooth manifold, with its Euclidean (standard) topology. Cylindrical coordinates are denoted by $(r, \theta; z)$, with (r, θ) polar coordinates in $\mathbb{R}^2$. For each of the following 1-forms $\eta \in \Omega^1(\mathbb{R}^3)$:

- Qualitatively describe and draw the distribution $\ker(\eta) \subseteq T\mathbb{R}^3$.[1]
- Decide whether $\ker(\eta)$ is a 2-plane distribution. In case it is, find two vector fields V_1, V_2 in $\mathbb{R}^3$ such that $\ker(\eta) = \langle V_1, V_2 \rangle$.
- Decide whether $\ker(\eta)$ is a contact structure, a foliation, or neither.

(a) $\eta = dz$, (b) $\eta = dz - ydx$, (c) $\eta = ydx$, (d) $\eta = dz - (ydx - xdy)$

(e) $\eta = dz - r^2 d\theta$, (f) $\eta = dz - (ydx + xdy)$, (g) $\eta = xdx + ydy + zdz$.

[1] You may draw $\ker(\eta)$ at a few points of $\mathbb{R}^3$, for instance. Also, you may explain in words your drawing, or use a computer if you prefer. Your description should be precise and clear enough so that somebody who has never imagined $\ker(\eta)$ can actually imagine it.

Problem 2. Let $(\mathbb{R}^3, \xi_{\mathrm{st}})$ be the standard contact structure $\xi_{\mathrm{st}} = \ker\{dz - ydx\}$. Let $p, q \in \mathbb{R}^3$ be two arbitrary but fixed points, with $p \neq q$.

(a) Show that there exists a smooth embedded path $\gamma : [0,1] \longrightarrow \mathbb{R}^3$ such that $\gamma(0) = p$, $\gamma(1) = q$, i.e. the path starts at p and ends at q, and $T\gamma([0,1]) \subseteq \xi_{\mathrm{st}}$, i.e. the tangent space of the path is always contained in ξ_{st}.

(b) Show that there exists a smooth embedded path $\tau : [0,1] \longrightarrow \mathbb{R}^3$ such that $\tau(0) = p$, $\tau(1) = q$, and $T\tau([0,1]) \neq\subseteq \xi_{\mathrm{st}}$, i.e. the tangent space of the path is never contained in ξ_{st}.

(c) Do parts (a) and (b) still hold if $p = q$?

(d) Consider the 2-plane distribution $\zeta \subseteq T\mathbb{R}^3$ given by $\zeta = \ker\{dz\}$. Do parts (a) and (b) still hold if we consider ζ instead of ξ?

Note that a corollary of (a) is that you can always parallel park.

Problem 3. Let $(\mathbb{R}^3, \xi_{\mathrm{st}})$ be the standard contact structure $\xi_{\mathrm{st}} = \ker\{dz - ydx\}$. For each of the following diffeomorphisms $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, discuss whether ϕ preserves ξ_{st} or not. That is, discuss whether $\phi_*\xi_{\mathrm{st}} = \xi_{\mathrm{st}}$. You may do this by drawing $\phi_*\xi_{\mathrm{st}}$ and ξ_{st}, or by computing with contact 1-forms[2], or in any other manner you wish.

(a) $\phi(x,y,z) = (x+3, y, z)$, (b) $\phi(x,y,z) = (x, y+4.3, z)$,

(c) $\phi(x,y,z) = (x, y, z-0.9)$, (d) $\phi(x,y,z) = (x+\cos(y), y, z+\cos(y))$,

(e) $\phi(x,y,z) = (x, y+f(x), z+g(x))$, where $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ satisfy $g'(x) = f(x)$,

(f) $\phi(x,y,z) = (x, y+ze^x, e^x z)$, (g) $\phi(x,y,z) = (x, y+z, (\cos(x)+2)z)$,

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Problem 4. Let $(\mathbb{R}^3, \xi_{\mathrm{st}})$ be the standard contact structure $\xi_{\mathrm{st}} = \ker\{dz - ydx\}$ and consider the two projections $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\pi(x,y,z) = (x,z)$, and $\Pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, $\Pi(x,y,z) = (x,y)$.

(a) Show that the fibers of π, i.e. each subset $\pi^{-1}(t)$, for any $t \in \mathbb{R}^2$, is a 1-dimensional embedded curve in $\mathbb{R}^3$ whose tangent space is contained in ξ_{st}.[3]

(b) Is part (a) true for the projection Π?

(c) Consider the embedded curve $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ parametrized by

$$\gamma(t) = (x(t), y(t), z(t)) = (-\cos(t), -3\sin(t)\cos(t), -\sin^3(t)), \quad t \in \mathbb{S}^1.$$

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[2]Note that, if you choose an equation α for ξ_{st}, so that $\xi_{\mathrm{st}} = \ker(\alpha)$, this is equivalent to the condition $\ker(\phi^*(\alpha)) = \ker(\alpha)$, i.e, the 1-form $\phi^*(\alpha)$ is a non-zero multiple of the 1-form α.

[3]This is the reason π is called a Legendrian projection.

Show that the image $\Gamma := \gamma(\mathbb{S}^1) \subseteq \mathbb{R}^3$ is an embedded 1-sphere $\mathbb{S}^1$ in $\mathbb{R}^3$, and prove that $T\Gamma \subseteq \xi_{\text{st}}$, i.e. its tangent space is always contained in ξ_{st}.

(d) Draw $\Gamma \subseteq \mathbb{R}^3$ in 3-space and draw the images $\pi(\Gamma), \Pi(\Gamma)$ of Γ under the two projections above, $\pi(\Gamma) \subseteq \mathbb{R}^2_{x,z}$ and $\Pi(\Gamma) \subseteq \mathbb{R}^2_{x,y}$.

(e) Consider an embedded curve $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ parametrized by

$$\gamma(t) = (x(t), y(t), z(t)), \quad t \in \mathbb{S}^1.$$

Find a necessary and sufficient condition for the functions $x(t), y(t), z(t)$ so that $\gamma(\mathbb{S}^1) \subseteq \mathbb{R}^3$ is everywhere tangent to ξ_{st}.

(f) (Optional) Let $K \subseteq \mathbb{R}^3$ be a knot such that $TK \subseteq \xi_{\text{st}}$, i.e. an embedding of the 1-sphere $\mathbb{S}^1$ into $\mathbb{R}^3$ whose tangent space is always contained in ξ_{st}.[4] Show that its image $\pi(K) \subseteq \mathbb{R}^2$ cannot be an embedded smooth curve in the plane $\mathbb{R}^2_{x,z}$.

(*Hint: How does one recover K from $\pi(K)$ if K is Legendrian?*)

Problem 5. (Frobenius Integrability Theorem) This problem discusses the following:

Theorem (Frobenius)**.** *Let $\zeta \subseteq T\mathbb{R}^3$ be a 2-plane distribution and $p \in \mathbb{R}^3$ a point. Suppose that $\eta \in \Omega^1(\mathbb{R}^3)$ is such that $\zeta = \ker(\eta)$. Then*

$$\zeta \textit{ is integrable near } p \quad \Longleftrightarrow \quad \eta \wedge d\eta = 0 \textit{ at } p.$$

By definition, ζ is said to be integrable near p if there exists an embedded smooth surface $\Sigma \subseteq \mathcal{O}p\,(p)$ in an open neighborhood $\mathcal{O}p\,(p)$ of p such that $T\Sigma = \zeta$. That is, at every point, the tangent space $T_p\Sigma$ coincides with the 2-plane field ζ.

Let us proceed in the following steps:

(a) Let V_1, V_2 be two vector fields in $\mathbb{R}^3$ and $\phi^t_{V_1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\phi^t_{V_2} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ their time-t flows. Show that their flows commute[5] if and only if $[V_1, V_2] = 0$.

Here $[V_1, V_2]$ denotes the Lie bracket on the vector fields. I.e. if we write

$$V_1 = f_1\partial_x + f_2\partial_y + f_3\partial_z, \quad V_2 = g_1\partial_x + g_2\partial_y + g_3\partial_z,$$

with $f_i, g_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$, then the Lie bracket $[V_1, V_2]$ is the vector field

$$[V_1, V_2] = \sum_{i=1}^{3}\sum_{i=j}^{3}(f_j \cdot (\partial_j g_i) - g_j \cdot (\partial_j f_i)) \cdot \partial_i,$$

where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$, and $\partial_3 = \partial_z$ are the three standard partial derivatives.

(b) Let V_1, V_2 be such that $\zeta = \langle V_1, V_2 \rangle$, i.e. V_1, V_2 span the distribution ζ. At a point $p \in \mathbb{R}^3$, show that $[V_1, V_2] \in \zeta$ if and only if $\eta \wedge d\eta = 0$.

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[4] This is referred to as a *Legendrian knot* $K \subseteq (\mathbb{R}^3, \xi_{\text{st}})$.

[5] I.e. $\phi^t_{V_1} \circ \phi^t_{V_2} = \phi^t_{V_2} \circ \phi^t_{V_1}$ as diffeomorphisms of $\mathbb{R}^3$.

(c) ($\Longrightarrow$) Show that
$$\zeta \text{ is integrable near } p \quad \Longrightarrow \quad [V_1, V_2] \in \zeta \text{ for any } V_1, V_2 \in \zeta.$$
(d) Prove the implication ($\Longleftarrow$) in the Frobenius theorem.