MAT 280: PROBLEM SET 3

DUE TO FRIDAY OCT 15 AT 9:00PM

ABSTRACT. This is the third problem set for the graduate course Contact and Symplectic Topology in the Fall Quarter 2021. It was posted online on Thursday Oct 7 and is due Friday Oct 15 at 9:00pm via online submission.

Task and Grade: Solve one of the five problems Problem 1 through Problem 5 below.[1]The maximum possible grade is 100 points. Despite the task being one problem, I strongly encourage you to work on the five problems.

Instructions: It is good to consult with other students and collaborate when working on the problems. You should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. By convention, all fronts are oriented by choose the highest point and adding an arrow to the right.5

Problem 1. Let $K \subseteq \mathbb{R}^3$ be a smooth knot whose projection onto the (x, z)-plane is a regular knot diagram.

(i) Show that there exists a Legendrian $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ which is C^0-close to K.

That is, for any $\varepsilon \in \mathbb{R}^+$, there exists a Legendrian $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ in the same smooth type of K such that $\Lambda \subseteq N_\varepsilon(K)$, where $N_\varepsilon(K)$ denote a tubular neighborhood of size ε, i.e. $N_\varepsilon(K) \cong K \times \mathbb{D}^2(\varepsilon)$ with $\mathbb{D}^2(\varepsilon)$ a disk of radius ε.

(ii) Show that there exists no Legendrian $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ which is C^1-close to K.

Remark: C^1-close means that both Λ and its tangent spaces can be made arbitrarily close to K and its tangent spaces.

Problem 2. Let $(\mathbb{R}^3, \xi_{\text{st}})$ be the standard contact structure $\xi_{\text{st}} = \ker\{dz - ydx\}$. Given a Legendrian front $\Gamma \subseteq \mathbb{R}^2_{x,z}$, consider a Legendrian lift $\Lambda_\Gamma \subseteq (\mathbb{R}^3, \xi_{\text{st}})$.

(i) For each of the Legendrian fronts $\Gamma \subseteq \mathbb{R}^2_{x,z}$ in Figure 1 determine the Thurston-Bennequin invariant tb and rotation number r of the associated Legendrian knot $\Lambda_\Gamma \subseteq (\mathbb{R}^3, \xi_{\text{st}})$. If a Legendrian front is associated to a Legendrian link, compute (tb, r) for each of the components.

[1]There is also a practice problem if you want to think about smooth knots, but it is not graded.

(ii) Consider the Legendrian knots in Figure 1.(1) and (2). Show that these two Legendrian knots are *not* Legendrian isotopic, and also that their underlying smooth knots are *smoothly* isotopic.

Remark: This implies that locally adding a zig-zag is not a Reidemeister move.

(iii) Consider the set of pairs of numbers

$$\mathcal{F} := \{(-n, m) \in \mathbb{N} \times \mathbb{N} : n \geq 1, m \in [-n+1, n-1]\}.$$

For each $(a,b) \in \mathcal{F}$, show that there exists a Legendrian representative $\Lambda_0^{(a,b)} \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ of the smooth unknot with associated formal invariants $(tb, r) = (a, b)$.

(iv) (Optional and Hard) Show that there exists no Legendrian representative $\Lambda_0 \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ of the smooth unknot with $tb(\Lambda_0) = 0$.

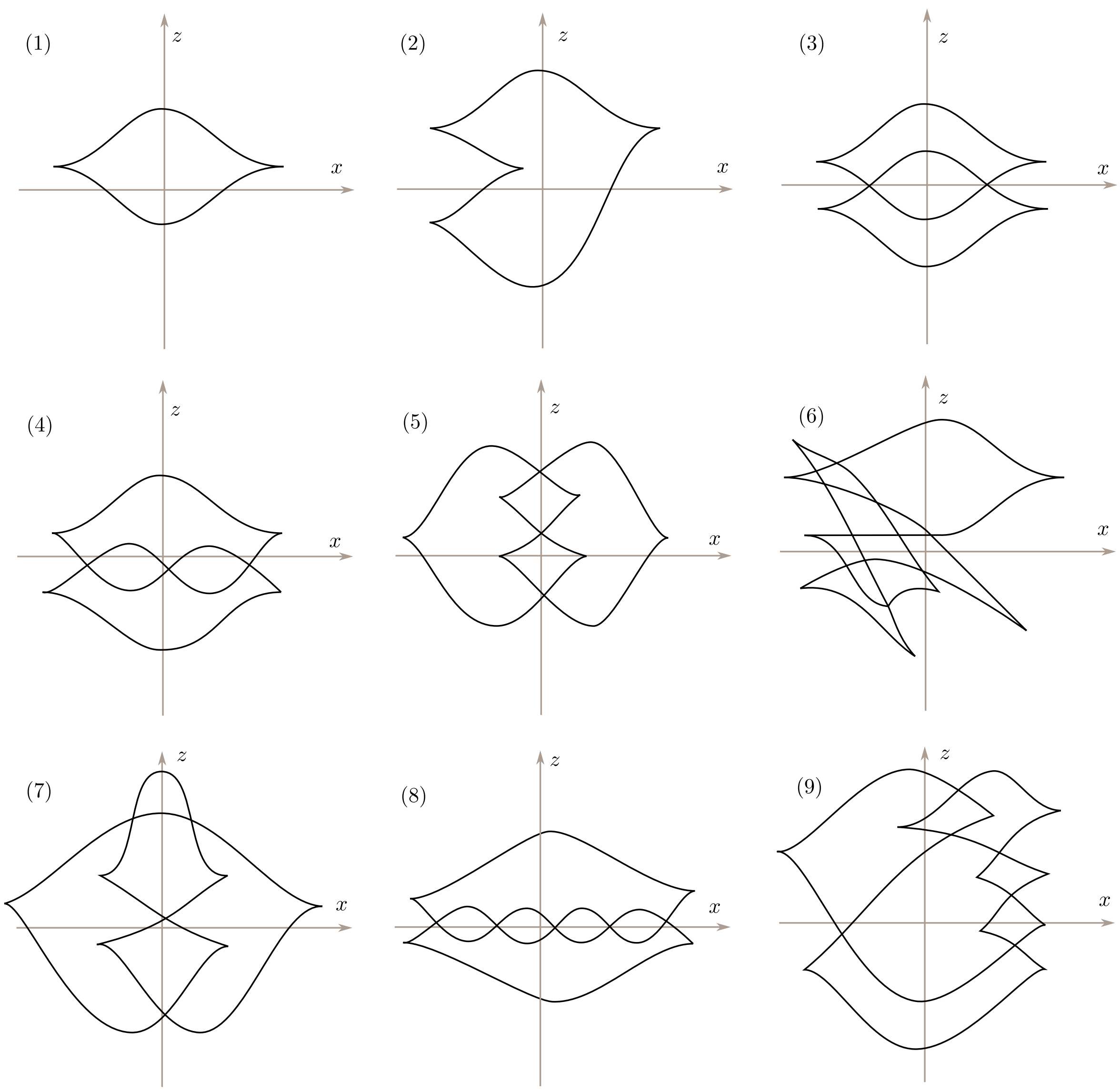

FIGURE 1. Nine Legendrian fronts for Problem 1.

Problem 3. Consider the two Legendrian knots $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \xi_{st})$ in Figure 2.

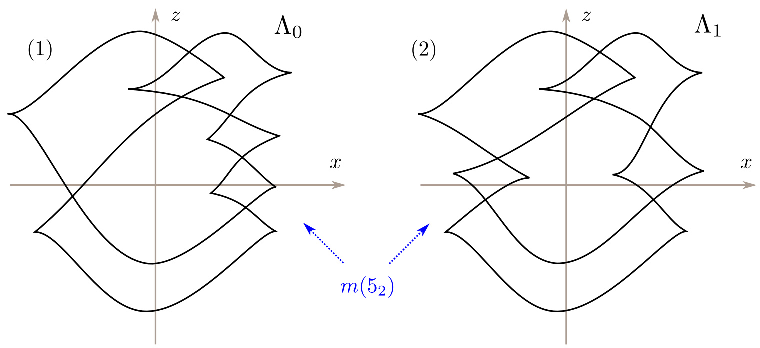

FIGURE 2. Legendrian representatives of the $m(5_2)$ smooth knot, one of the two knots with five crossings.

(i) Show that the smooth knots underlying Λ_0, Λ_1 are smoothly isotopic and are in the smooth isotopy class $m_(5_2)$ of the mirror of the 5_2 knot.

(ii) Compute the formal invariants (tb, r) of both Λ_0 and Λ_1.

(iii) Show that the positive stabilizations $S_+(\Lambda_0), S_+(\Lambda_1)$ are Legendrian isotopic.

(iv) Show that the negative stabilizations $S_-(\Lambda_0), S_-(\Lambda_1)$ are Legendrian isotopic.

(v) (Optional and hard) Show that Λ_0 and Λ_1 are **not** Legendrian isotopic.

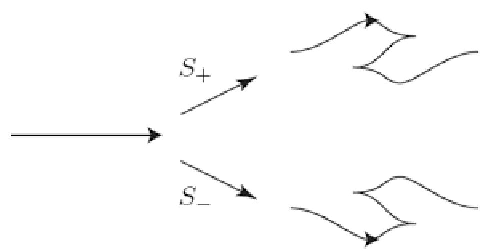

FIGURE 3. The positive stabilization S_+ operation on a front, and the negative stabilization S_- operation.

Problem 4. Solve the following parts:

(i) Show that tb and r are invariants of the Legendrian isotopy class, i.e. if $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \xi)$ are Legendrian isotopic, then $(tb(\Lambda_0), r(\Lambda_0)) = (tb(\Lambda_1), r(\Lambda_1))$.

Hint: You may use the Legendrian Reidemeister theorem, see Figure 4.

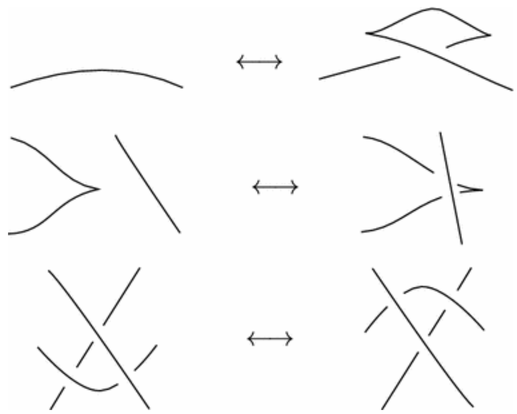

FIGURE 4. The three Legendrian Reidemeister moves.

(ii) Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ be a Legendrian knot, and $S_+(\Lambda)$ and $S_-(\Lambda)$ its positive and negative stabilizations, as depicted in Figure 3. Compute the formal invariants (tb, r) of $S_+(\Lambda)$ and $S_-(\Lambda)$ in terms of those of Λ.

(iii) Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{st})$ be a Legendrian knot with front $\pi(\Lambda) \subseteq \mathbb{R}^2$ and two smooth points $p, q \in \pi(\Lambda)$. Consider the front $S_+(\Lambda, p)$ obtained by performing a positive stabilization to $\pi(\Lambda)$ in a neighborhood of p, and the front $S_+(\Lambda, q)$ obtained by performing a positive stabilization to $\pi(\Lambda)$ in a neighborhood of q. Show that $S_+(\Lambda, p)$ and $S_+(\Lambda, q)$ are Legendrian isotopic, i.e. the Legendrian isotopy class of a positive stabilization remains the same regardless of where it is performed.

Remark: the same is true for a negative stabilization.

(iv) Suppose that $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \xi_{st})$ are two Legendrian knots whose underlying smooth type coincides. (They might not be Legendrian isotopic.) Consider the Legendrian $S_+^k S_-^l(\Lambda_0)$ whose front is $\pi(\Lambda_0)$ with k positive stabilizations added and l negative stabilizations added, $k, l \in \mathbb{N}$. Similarly define $S_+^m S_-^n(\Lambda_1)$ for $m, n \in \mathbb{N}$.

Show that for any $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \xi_{st})$ as above, there exist $k, l, m, n \in \mathbb{N}$ such that $S_+^k S_-^l(\Lambda_0)$ is Legendrian isotopic to $S_+^m S_-^n(\Lambda_1)$.

Hint: The smooth types of Λ_0 and Λ_1 coincide, thus there exist a sequence of *smooth* Reidemeister moves between their smooth knot diagrams. Use the additional stabilizations to realize this smooth isotopy as a Legendrian isotopy.

Problem 5. (Geometric meaning of tb and r) Let $\Lambda \subseteq (\mathbb{R}^3, \xi_{\text{st}})$ be an oriented Legendrian knot.

(i) Let V be a vector field along Λ which is transverse to ξ, i.e. $V \notin \xi$. Consider the Legendrian Λ' obtained by pushing Λ along this vector field V. Show that the Thurston-Bennequin invariant $tb(\Lambda) = \text{lk}(\Lambda, \Lambda')$ is equal to the linking number $\text{lk}(\Lambda, \Lambda')$ between Λ and Λ'.

(ii) Consider an oriented surface $S \subseteq \mathbb{R}^3$ such that $\partial S = \Lambda$ as oriented subspaces. Show that the restriction of the real rank-2 bundle $\xi|_S \longrightarrow S$ defines a trivial vector bundle, i.e. there exists an isomorphism $\xi|_S \cong S \times \mathbb{R}^2$ as oriented real rank-2 bundles.

(iii) Let $\gamma : \mathbb{S}^1 \longrightarrow \Lambda \subseteq \mathbb{R}^3$ be a regular parametrization of Λ compatible with its orientation. By using the trivialization $\xi|_S \cong S \times \mathbb{R}^2$ in (ii), its derivative $\gamma' : \mathbb{S}^1 \longrightarrow \xi|_\Lambda$ defines a map $\gamma' : \mathbb{S}^1 \longrightarrow \mathbb{R}^2 \setminus \{0\}$. Show that $r(\Lambda) = \deg(\gamma')$.

Practice problem. This is a practice problem on smooth knots, there is no contact geometry involved. Let $\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the projection $\pi(x, y, z) = (x, z)$.

(i) Show that any knot $K \subseteq \mathbb{R}^3$ is smoothly isotopic to a knot whose image under the π-projection is a *regular* knot diagram.

(ii) Prove the smooth Reidemeister Theorem.