MAT 280: PROBLEM SET 5

DUE TO FRIDAY OCT 29 AT 9:00PM

Abstract. This is the fifth problem set for the graduate course Contact and Symplectic Topology in the Fall Quarter 2021. It was posted online on Friday Oct 22 and is due Friday Oct 29 at 9:00pm via online submission.

Task and Grade: Solve one of the five problems Problem 1 through Problem 5 below. The maximum possible grade is 100 points. Despite the task being one problem, I strongly encourage you to work on the five problems. Problem 0 is for practice.

Instructions: It is good to consult with other students and collaborate when working on the problems. You should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. By convention, all fronts are oriented by choose the highest point and adding an arrow to the right.

Notation: Given two smooth manifolds X, Y, a smooth map $f : X \longrightarrow Y$, and a sheaf $\mathscr{F} \in \mathrm{Sh}(X)$. The direct image sheaf $f_*\mathscr{F} \in \mathrm{Sh}(Y)$ is defined by the pre-sheaf $f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$. In this Problem Set, *computing sheaf cohomology* means computing the $\mathbb{C}$-vector spaces $H^*(X, \mathscr{F})$ for all $* \in \mathbb{N}$.

Problem 0. In this problem, you may take $X = Y = \mathbb{R}$ if you would like. Solve the following two parts:

(a) Let $f : X \longrightarrow Y$ be a smooth map between smooth manifolds, and $\mathscr{F} \in \mathrm{Sh}(X)$ be a sheaf. Show that the direct image pre-sheaf $f_*\mathscr{F} \in \mathrm{PreSh}(Y)$, defined by $f_*\mathscr{F}(V) := \mathscr{F}(f^{-1}(V))$, for any open set $V \subseteq Y$, and the restriction maps induced from $\mathscr{F}$, is in fact sheaf $f_*\mathscr{F} \in \mathrm{Sh}(Y)$.

(b) Let $\mathscr{F}_1, \mathscr{F}_2 \in \mathrm{Sh}(X)$ be sheaves of $\mathbb{C}$-vector spaces in a smooth manifold X. Show that the pre-sheaf $\mathscr{F}_1 \oplus \mathscr{F}_2 \in \mathrm{PreSh(X)}$ defined by $(\mathscr{F}_1 \oplus \mathscr{F}_2)(U) := \mathscr{F}_1(U) \oplus \mathscr{F}_2(U)$ for any open set $U \subseteq X$, and the natural component-wise restriction maps, is in fact a sheaf $\mathscr{F}_1 \oplus \mathscr{F}_2 \in \mathrm{Sh(X)}$.

Problem 1. Let $X = \mathbb{R}^2$ be the real plane. For each of the following pre-sheaves $\mathscr{F}$ of $\mathbb{C}$-vector spaces on $\mathbb{R}^2$, determine whether it is a sheaf. (Prove your claims.)

(i) $\mathscr{F}$ is the pre-sheaf of constant $\mathbb{C}$-valued functions, i.e. the sheaf defined by

$$\mathscr{F}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ constant}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions.

(ii) $\mathscr{F}$ is the pre-sheaf of locally constant $\mathbb{C}$-valued functions, defined by

$$\mathscr{F}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ locally constant}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions.

(iii) $\mathscr{F}$ is the pre-sheaf of smooth $\mathbb{C}$-valued functions, i.e. the sheaf defined by

$$\mathscr{F}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ smooth}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions.

(iv) $\mathscr{F}$ is the pre-sheaf of bounded smooth $\mathbb{C}$-valued functions, i.e. defined by

$$\mathscr{F}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ bounded and smooth}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions.

(v) $\mathscr{F}$ the pre-sheaf of smooth $\mathbb{C}$-valued functions with prescribed values as follows:

$$\mathscr{F}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ smooth and } f(0,0) = 0, f(1,1) = 2\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions.

(vi) $\mathscr{F}$ is the pre-sheaf of constant $\mathbb{C}$-valued functions, i.e. the sheaf defined by

$$\mathscr{F}(U) := \{X : U \longrightarrow TU \text{ a section}, i.e.\ X \text{ is vector field on U}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with morphisms given by restricting vector fields.

(vii) (Optional) Endow $\mathbb{R}^2 = \mathbb{C}$ with the standard complex structure, and consider the pre-sheaf of holomorphic functions:

$$\mathscr{F}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ holomorphic}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions.

Problem 2. Let X be a smooth manifold and $\underline{\mathbb{C}} \in Sh(X)$ be the constant sheaf $\mathscr{F} = \underline{\mathbb{C}}$ sheaf of $\mathbb{C}$-vector spaces given by

$$\underline{\mathbb{C}}(U) := \{f : U \longrightarrow \mathbb{C} : f \text{ locally constant}\},$$

for open sets $U \subseteq \mathbb{R}^2$, with restriction morphisms given by restricting functions. For each of the following X, compute the sheaf cohomology $H^*(X, \mathscr{F})$ of the sheaf $\mathscr{F} = \underline{\mathbb{C}}$:

(a) $X = \mathbb{R}$, (b) $X = \mathbb{R}^2$, (c) The 1-sphere $X = S^1$, (d) The 2-sphere $X = S^2$

(e) The 2-torus $X = S^1 \times S^1$, (f) (Optional) $X = \mathbb{R}^n$, (g) (Optional) $X = S^n$.

Problem 3. For each of the following pairs $(X, \mathscr{F})$ of a smooth manifold X and a sheaf $\mathscr{F} \in \mathrm{Sh}(X)$, compute the stated sheaf cohomology.

(i) Let $i : \{pt\} \longrightarrow \mathbb{R}^2$ be the inclusion of the origin $i(pt) = (0, 0) \in \mathbb{R}^2$ and $\mathscr{F} = \underline{\mathbb{C}} \in \mathrm{Sh}(pt)$ the constant sheaf at the point $\{pt\}$. Compute $H^*(\mathbb{R}^2, i_*\mathscr{F})$.

(ii) Let $i : Z \longrightarrow \mathbb{R}^3$ be the inclusion of the ellipsoid
$$Z = \{(x, y, z) : 3x^2 + 2y^2 + 4.1z^2 = 1\}$$
and $\mathscr{F} = \underline{\mathbb{C}} \in \mathrm{Sh}(Z)$ the constant sheaf at $Z = \mathbb{S}^2$. Compute $H^*(\mathbb{R}^3, i_*\mathscr{F})$.

(iii) Let $i : Z \longrightarrow \mathbb{R}^2$ be the inclusion of the ellipsoid
$$Z = \{(x, y, z) : 3x^2 + 2y^2 + 4.1 \cdot z^2 = 1\}$$
and consider the sheaf $\mathscr{F} = \mathscr{C}^\infty_Z \in \mathrm{Sh}(S^1)$ of smooth functions on $Z = \mathbb{S}^1$. Compute $H^*(\mathbb{R}^2, i_*\mathscr{C}^\infty_{S^1})$.

(iv) Let $i_k : \mathbb{R} \times [0, \infty) \longrightarrow \mathbb{R}^2$ denote the inclusion $i_k(x, y) = (x, y + k)$. Consider the sheaf $\mathscr{F} = ((i_1)_*\mathbb{C}) \oplus ((i_2)_*\mathbb{C})$. Compute $H^*(\mathbb{R}^2, \mathscr{F})$.

Problem 4. (A taste of the sheaf viewpoint on differential equations)

(i) Consider the real differential equation $x'(t) = 3x(t)$ with $x(t) \in \mathbb{R}$, and its pre-sheaf of solutions $\mathscr{F} \in \mathrm{PreSh}(\mathbb{R})$, defined by
$$\mathscr{F}(U) = \{x : U \longrightarrow \mathbb{R} : x'(t) = 3x(t)\}, \quad U \subseteq \mathbb{R}_t.$$
Show that $\mathscr{F}$ is a sheaf $\mathscr{F} \in \mathrm{Sh}(\mathbb{R})$ of $\mathbb{R}$-vector spaces on the real line $\mathbb{R}$ parametrizing the time $t \in \mathbb{R}$. What is the stalk $\mathscr{F}_p$ at a point $p \in \mathbb{R}$?

(ii) Consider the real differential equation $x'(t) = x(t)^2$ with $x(t) \in \mathbb{R}$, and its pre-sheaf of solutions $\mathscr{F} \in \mathrm{PreSh}(\mathbb{R})$, defined by
$$\mathscr{F}(U) = \{x : U \longrightarrow \mathbb{R} : x'(t) = x(t)^2\}.$$
Show that $\mathscr{F}$ is a sheaf $\mathscr{F} \in \mathrm{Sh}(\mathbb{R})$ of sets. Is it a sheaf of $\mathbb{R}$-vector spaces?

(iii) Let $y'(z) = Ay(z)$ be a complex linear differential equation, $y(z) \in \mathbb{C}^n$, $z \in \mathbb{C}$, $A \in M_{n\times n}(\mathbb{C})$. Show that its pre-sheaf of solutions is a sheaf of $\mathbb{C}$-vector spaces on $\mathbb{C} = \mathbb{C}_z$. Compute its stalks at each point on $\mathbb{C}$, parametrizing the (complex) time variable $z \in \mathbb{C}$.

(iv) Consider the complex differential equation $zy'(z) = \sigma \cdot y(z)$, where $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $y : \mathbb{C}^* \longrightarrow \mathbb{C}$ and $\sigma \in \mathbb{C}$ a parameter. Compute the global sections of its sheaf of solutions.

Hint: the space of global sections might depend on the parameter $\sigma \in \mathbb{C}$.

(v) (Optional) Consider the map $f_n : \mathbb{C}_z \longrightarrow \mathbb{C}$, $f(z) = z^n$, $z \in \mathbb{C}$ and the push-forward sheaf $(f_n)_*\underline{\mathbb{C}} \in \mathrm{Sh}(\mathbb{C})$. Compute the stalks of $(f_n)_*\underline{\mathbb{C}} \in \mathrm{Sh}(\mathbb{C})$ at every point. Find the cohomology groups $H^0(\mathbb{C}^*, (f_n)_*\underline{\mathbb{C}}) \in \mathrm{Sh}(\mathbb{C}))$ and $H^1(\mathbb{C}^*, (f_n)_*\underline{\mathbb{C}}) \in \mathrm{Sh}(\mathbb{C}))$ of $(f_n)_*\underline{\mathbb{C}}$ restricted to $\mathbb{C}^* \subseteq \mathbb{C}$.

Problem 5. Let $f_x^{(1)}:\mathbb{R}_\tau \longrightarrow \mathbb{R}_z$ be a generating family for the front in Figure 1.(1), $f_x^{(2)}:\mathbb{R}_\tau \longrightarrow \mathbb{R}_z$ a generating family for Figure 1.(2), and $f_x^{(3)}:\mathbb{R}_\tau \longrightarrow \mathbb{R}_z$ a generating family for Figure 1.(3). For each of the three fronts in Figure 1:

(i) For each open set $U_i \subseteq \mathbb{R}^2_{x,z}$ on the front, as depicted in Figure 1, compute the sheaf cohomology groups

$$H^*(\pi^{-1}(U_i)\cap\{(x,z,\tau): f_x(\tau)\le z\},\underline{\mathbb{C}}).$$

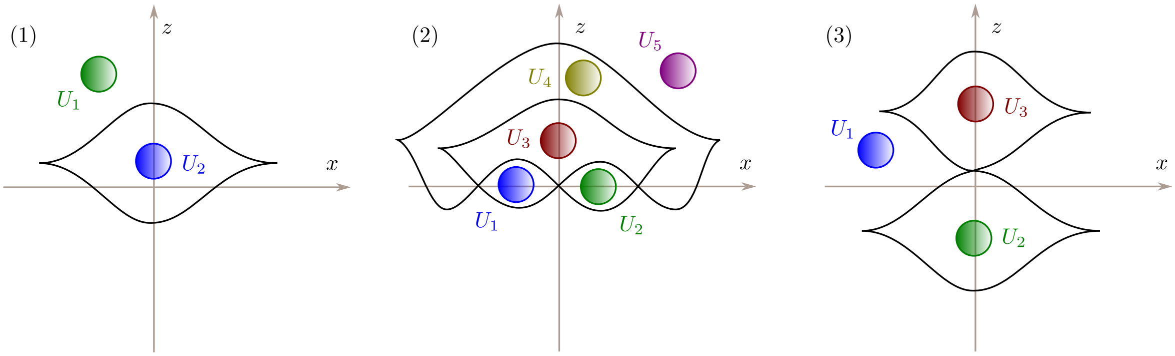

Figure 1. Fronts and open sets for Problem 5.

(ii) (Optional) Let $i: Z \longrightarrow \mathbb{R}^3$ be the inclusion of the sublevel sets

$$Z := \{(x,z,\tau): f_x(\tau)\le z\}\subseteq \mathbb{R}^3,$$

and the sheaf $i_*\underline{\mathbb{C}} \in \mathbb{R}^3$. Compute the stalks of the higher direct images $R^i\pi_*(i_*\underline{\mathbb{C}}) \in \mathrm{Sh}(\mathbb{R}^2)$ on the complement of the front.