MAT 280: PROBLEM SET 6

DUE TO FRIDAY NOV 12 AT 9:00PM

Abstract. This is the sixth problem set for the graduate course Contact and Symplectic Topology in the Fall Quarter 2021. It was posted online on Saturday Nov 6 and is due Friday Nov 12 at 9:00pm via online submission.

Task and Grade: Solve one of the five problems Problem 1 through Problem 5 below. The maximum possible grade is 100 points. Despite the task being one problem, I strongly encourage you to work on the five problems.

Instructions: It is good to consult with other students and collaborate when working on the problems. You should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page. By convention, all fronts are oriented by choose the highest point and adding an arrow to the right.

Notation: Given a Legendrian $\Lambda \subseteq (\partial^\infty(T^*\mathbb{R}^2), \xi_{\text{st}})$, we denote by $\mathfrak{M}(\Lambda)$ the space of constructible sheaves in $\mathbb{R}^2$ (of chain complexes of $\mathbb{C}$-vector spaces) whose singular support at infinity is contained in Λ. The subset $\mathfrak{M}_1(\Lambda) \subseteq \mathfrak{M}(\Lambda)$ is cut out by those sheaves with microlocal rank 1 and whose stalk at any unbounded region of $\mathbb{R}^2$, for any finite stratification for which they are constructible, is acyclic.

Problem 1. For each of the following sheaves $\mathscr{F} \in \text{Sh}(\mathbb{R})$ of $\mathbb{C}$-vector spaces, compute the singular support $\mu supp(\mathscr{F}) \subseteq T^*\mathbb{R}$.

(i) Consider the closed inclusion $i : Z \longrightarrow \mathbb{R}$ of the closed interval $Z = [-2, 3]$ into the real line and the direct image sheaf $\mathscr{F} = i_*\mathbb{C}_Z$ in $\text{Sh}(\mathbb{R})$.

(ii) Consider the inclusion $j : Z \longrightarrow \mathbb{R}$ of the open interval $U = (-1, 4)$ into the real line and the compactly supported direct image sheaf $\mathscr{F} = j_!\mathbb{C}_U$ in $\text{Sh}(\mathbb{R})$.

(iii) Consider the inclusion $k : A \longrightarrow \mathbb{R}$ of the interval $A = [-2, 3)$ into the real line and the compactly supported direct image sheaf $\mathscr{F} = k_!\mathbb{C}_A$ in $\text{Sh}(\mathbb{R})$.

(iv) Let i, j be the inclusions in parts (i) and (ii), and consider the direct sum sheaf $\mathscr{F} = i_*\mathbb{C}_Z \oplus j_!\mathbb{C}_U \in \text{Sh}(\mathbb{R})$.

(v) Consider the sheaf $\mathscr{F} = C^\infty_{\mathbb{R}}$ of smooth functions on $\mathbb{R}$.

(vi) (Optional) Let $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $\pi(x, y) = x$ and let $i : Z \longrightarrow \mathbb{R}^2$ be given by the unit closed ball. Consider the sheaf $\mathscr{F} = \pi_*(i_*(Z))$.

Problem 2. For each of the following sheaves $\mathscr{F} \in \mathrm{Sh}(\mathbb{R}^2)$ of $\mathbb{C}$-vector spaces, compute the singular support $\mu supp(\mathscr{F}) \subseteq T^*\mathbb{R}^2$.

(a) Consider the closed inclusion $i : Z \longrightarrow \mathbb{R}^2$ of the unit ball
$$Z = \{p \in \mathbb{R}^2 : \|p\|^2 \leq 1\}$$
and the direct image sheaf $\mathscr{F} = i_*\mathbb{C}_Z$ in $\mathrm{Sh}(\mathbb{R}^2)$.

(b) Consider the closed inclusion $i : Z \longrightarrow \mathbb{R}^2$ of the unit square
$$Z = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_\infty \leq 1\} = \{(x, y) \in \mathbb{R}^2 : \max(|x|, |y|) \leq 1\}$$
and the direct image sheaf $\mathscr{F} = i_*\mathbb{C}_Z$ in $\mathrm{Sh}(\mathbb{R}^2)$.

(c) Consider the open inclusion $j : U \longrightarrow \mathbb{R}^2$ of the unit open ball
$$U = \{p \in \mathbb{R}^2 : \|p\|^2 < 1\}$$
and the sheaf $\mathscr{F} = j_!\mathbb{C}_U$.

(d) Consider the inclusion $j : U \longrightarrow \mathbb{R}^2$ of the unit open square
$$Z = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_\infty < 1\} = \{(x, y) \in \mathbb{R}^2 : \max(|x|, |y|) < 1\}$$
and the sheaf $\mathscr{F} = j_!\mathbb{C}_U$.

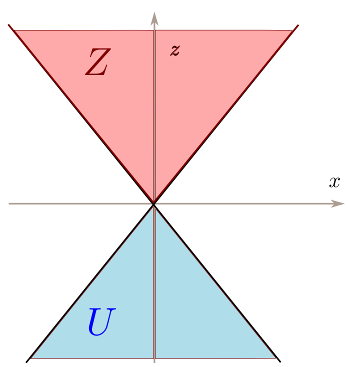

FIGURE 1. Open set U and closed set Z for Problem 2.

(e) Let $U, Z \subseteq \mathbb{R}^2$ be the open and closed sets depicted in Figure 1, with inclusions $j : U \longrightarrow \mathbb{R}^2$ and $i : Z \longrightarrow \mathbb{R}^2$. Consider the sheaf $\mathscr{F} = i_*\mathbb{C}_Z \oplus j_!\mathbb{C}_U$.

(f) (Optional) In the same notation as Part (e), let $\mathscr{F}$ be any non-trivial extension of $i_*\mathbb{C}_Z$ and $j_!\mathbb{C}_U$,[1], i.e. there exists a short exact sequence
$$0 \longrightarrow j_!\mathbb{C}_U \longrightarrow \mathscr{F} \longrightarrow i_*\mathbb{C}_Z \longrightarrow 0,$$
but $\mathscr{F} \neq i_*\mathbb{C}_Z \oplus j_!\mathbb{C}_U$.

(g) (Optional) Consider the sheaf of holomorphic functions $\mathcal{O}_\mathbb{C} \in \mathrm{Sh}(\mathbb{C})$ and let $\mathscr{D} : \mathcal{O}_\mathbb{C} \longrightarrow \mathcal{O}_\mathbb{C}$ be a differential operator. Relate the singular support of the complex $0 \longrightarrow \mathcal{O}_\mathbb{C} \xrightarrow{\mathscr{D}} \mathcal{O}_\mathbb{C} \longrightarrow 0$ to the principal symbol of $\mathscr{D}$.

[1]Part (e) is the case of the *trivial* extension in $\mathrm{Ext}^1(i_*\mathbb{C}_Z, j_!\mathbb{C}_U)$.

Problem 3. For each of the three Legendrian Reidemeister moves, let Λ_0 be the (standard) corresponding front before the Reidemeister move, and Λ_1 be the (standard) corresponding front after the Reidemeister move. Prove that $\mathfrak{M}(\Lambda_0)$ and $\mathfrak{M}(\Lambda_1)$ are canonically isomorphic.[2]

Problem 4. For each of the two fronts depicted in Figure 2.(i), let $\Lambda_i \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ be the associated Legendrian knot. Solve each of the following parts:

(i) Compute the spaces $\mathfrak{M}_1(\Lambda_1)$ and $\mathfrak{M}_1(\Lambda_2)$.

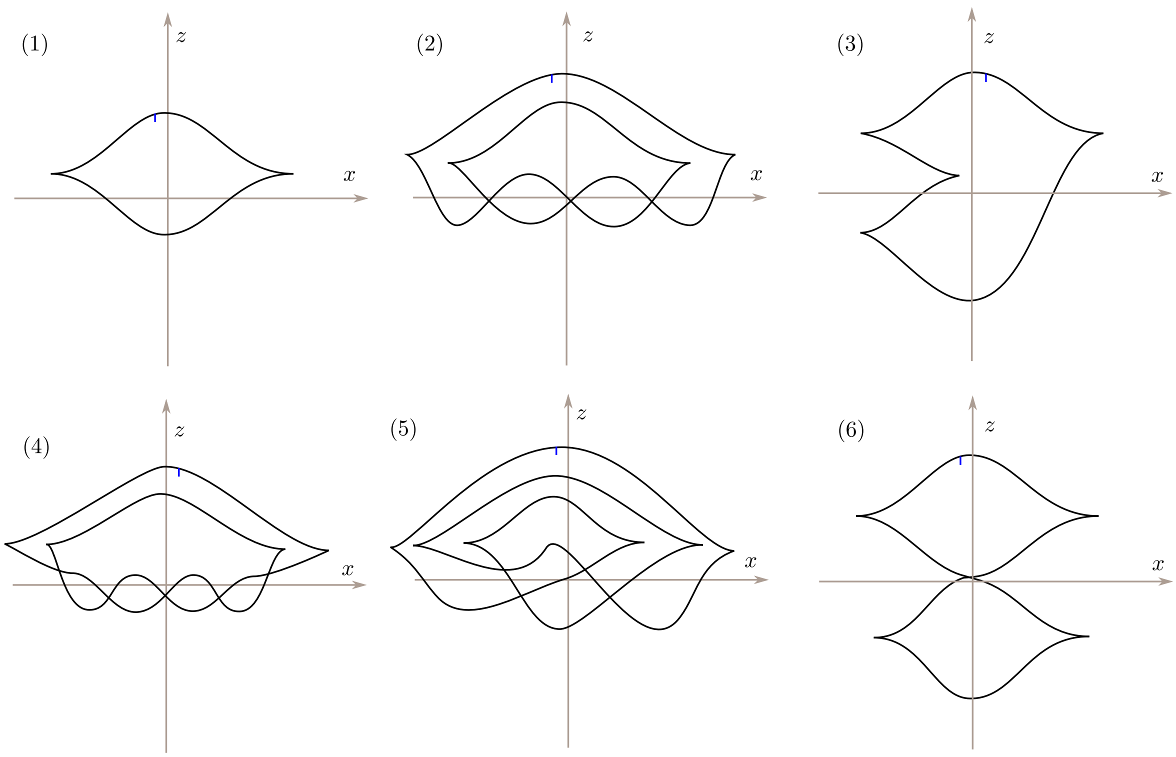

FIGURE 2. Fronts for Problem 4.

(ii) Show that the space $\mathfrak{M}_1(\Lambda_3)$ is empty.

(iii) Show that $\mathfrak{M}_1(\Lambda_4)$ is isomorphic to

$$X_7 := \{(p_1, p_2, \ldots, p_6, p_7) \in \mathbb{CP}^1 : p_i \neq p_{i+1} \text{ for } 1 \leq i \leq 6, \quad p_7 \neq p_1\},$$

where $\mathbb{CP}^1$ is the complex projective line.

(iv) Compute $\mathfrak{M}_1(\Lambda_5)$ and show that it is isomorphic to $\mathfrak{M}_1(\Lambda_2)$. Can you give a geometric reason why that is the case?

(v) (Optional) Compute $\mathfrak{M}_1(\Lambda_6)$ and show that it is isomorphic to $\mathfrak{M}_1(\Lambda_1)$.

(vi) Based on $\mathfrak{M}_1(\Lambda_2)$ and $\mathfrak{M}_1(\Lambda_4)$, compute $\mathfrak{M}_1(\Lambda(2,n))$ where $\Lambda(2,n) \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ is the maximal-tb representative of the $(2,n)$-torus link.

[2]In whichever setting you feel comfortable, e.g. sets, stacks, categories and so on.

Problem 5. Consider the two Legendrian knots $\Lambda_0, \Lambda_1 \subseteq (\mathbb{R}^3, \xi_{\mathrm{st}})$ associated to the two fronts depicted in Figure 3.

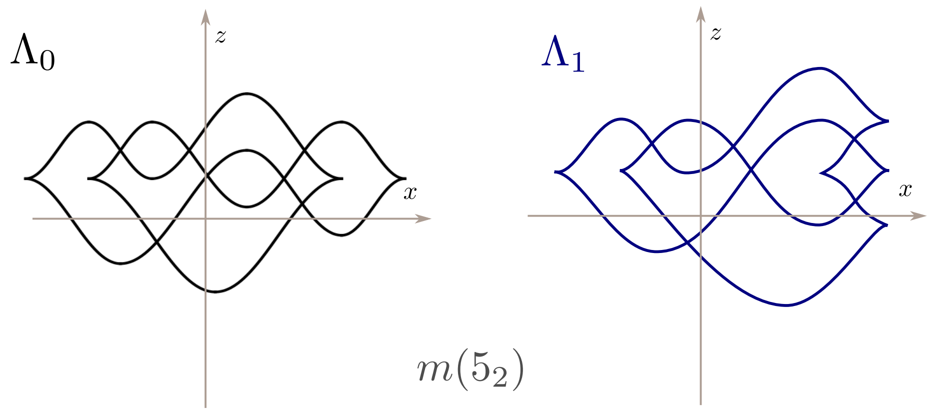

FIGURE 3. Fronts for the Chekankov pair, two Legendrian representatives of the knot $m(5_2)$ which are *not* Legendrian isotopic.

(a) Show that Λ_0 and Λ_1 are smoothly isotopic to the knot $m(5_2)$ and have the same Thurston-Bennequin and rotation numbers.

(b) Show that Λ_0 and Λ_1 are not Legendrian isotopic by computing the corresponding categories of constructibles sheaves in $\mathbb{R}^2$ with singular support in each of them and showing that they are not isomorphic.