review sequences: \( (a_n) = a_1, a_2, a_3, \ldots \)

* write terms

* decide

  - convergent: \( \lim_{n \to \infty} a_n \quad (\text{hierarchy growth, MCT, comparison test}) \)
  - divergent: \( a_n \to 0 \) or \( a_n \) (hierarchy growth, comparison test)

hierarchy growth: \( n(a(n)) \leq n^k \leq n^a \leq n! \leq n^n \)

MCT: increasing & bounded above / decreasing & bounded below

comparison test: \( b_n > a_n \quad \lim_{n \to \infty} b_n = \text{converges} \Rightarrow \lim_{n \to \infty} a_n = \text{converges} \)

ex) \( a_n = \frac{1}{2^n}, \quad n \geq 0 \)

1) write \( a_1, a_2, a_3, a_n, \ldots \)

\( a_0 = \frac{1}{a^n} = 1 \)

\( a_1 = \frac{1}{a}, \quad \frac{1}{a} \)

\( a_2 = \frac{1}{a^2} = \frac{1}{a^2} \)

\( a_3 = \frac{1}{a^3} = \frac{1}{a^3} \)

2) show \( (a_n) \) decreasing \( (a_{n+1} < a_n) \)

\( \frac{1}{a^{n+1}} < \frac{1}{a^n} \quad \to \quad 1 < \frac{a^{n+1}}{a^n} \quad \to \quad 1 < a \quad \checkmark \)

3) show \( a_n = \frac{1}{a^n} \) converges

1. compute directly

2. part 2 shows decreasing

also \( a_n = \frac{1}{a^n} \leq 0 \), bounded below

MCT = converges

* recursive -> possibly use 1 as value bounded by *

review series: \( \sum_{n=1}^{\infty} a_n \)

* decide convergent vs divergent

  - integral test
  - alternating series test: \( \sum (-1)^n a_n \)
  - ratio test: \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} \)
  - comparison test: \( \sum \frac{1}{n^{p+1}} \)

  - root test: \( \lim_{n \to \infty} \sqrt[n]{|a_n|} \)

  two special types:

  - \( p \)-series: \( \sum_{n=1}^{\infty} \frac{1}{n^p} \), \( p > 0 \) -> convergent if \( p > 1 \)

  - geometric series: \( \sum_{n=1}^{\infty} r^n = \frac{a}{1-r} \) -> if \( |r| < 1 \) — convergent

(review taylor series:

* taylor expansion: \( f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \)

  where \( a_n = \frac{f^{(n)}(0)}{n!} \)

ex) given \( f(x) = \sin(x) \) & \( a > 0 \)

\( f(0) = \sin(0) = 0 \)

\( f'(0) = \cos(0) = 1 \)
* Compute $f(x)$, $f'(x)$, $f''(x)$ ... (a typically $= 0$)

\[ a_n = \frac{f^{(n)}(a)}{n!} \]

- $f'(a) = \cos(a) \cdot 1$
- $f''(a) = -\sin(a) \cdot 0$
- $f'''(a) = -\cos(a) \cdot 1$

* Truncation at degree 5 $\rightarrow x^5$
* Taylor can multiply & plug in variables
* Taylor of polynomial = polynomial
* Memorize: $\sin(x)$, $\cos(x)$, $e^x$, $\ln(x)$