Abstract. Practice problems for the twelfth lecture of Part II, delivered June 2nd 2023. Solutions will be posted within 48h of these problems being posted.

Brief reminder from lecture: Given a critical point \((x_0, y_0)\) of a function \(f(x, y)\), the characteristic polynomial \(p(\lambda)\) at the critical point \((x_0, y_0)\) is computed as follows:

1. First, compute the four second derivatives
   \[ \partial_{xx} f, \partial_{yy} f, \partial_{xy} f, \partial_{yx} f. \]
   
   (Recall that you should always get \(\partial_{xy} f = \partial_{yx} f\).)

2. Second, evaluate the four second derivatives at \((x_0, y_0)\), setting:
   \[ a := \partial_{xx} f(x_0, y_0), \quad b := \partial_{xy} f(x_0, y_0), \]
   \[ c := \partial_{yx} f(x_0, y_0), \quad d := \partial_{yy} f(x_0, y_0). \]

3. The characteristic polynomial \(p(\lambda)\) at the critical point \((x_0, y_0)\) is the polynomial
   \[ \lambda^2 - (a + d)\lambda + (ad - bc). \]

The real roots \(\lambda_+, \lambda_-\) of the characteristic polynomial determine the type of critical point, as follows:

- If both \(\lambda_+, \lambda_- > 0\), then it is a minimum.
- If both \(\lambda_+, \lambda_- < 0\), then it is a maximum.
- If one of \(\lambda_+, \lambda_-\) is positive and the other is negative, then it is a saddle.
- Otherwise, i.e. if at least one of \(\lambda_+, \lambda_-\) is zero or the roots are imaginary, then we cannot decide.

Another fast way to check is to just compute \(a\) and \(ad - bc\) directly, then you can instead use the following criteria:

- If both \(ad - bc > 0\) and \(a > 0\), then it is a minimum.
- If \(ad - bc > 0\) and \(a < 0\), then it is a maximum.
- If \(ad - bc < 0\), then it is a saddle.
- If \(ad - bc = 0\), then we cannot decide.
**Problem 1.** Consider the function \( f(x, y) = x^2y^2 - 5x^2 - 5y^2 - 8xy. \)

(a) Show that the critical points are \((0, 0), (3, 3), (-3, -3), (1, -1)\) and \((-1, 1)\).

The first derivatives are \( f_x(x, y) = 2xy^2 - 10x - 8y \) and \( f_y(x, y) = 2x^2y - 10y - 8x \). Since \( f_x(0, 0) = f_y(0, 0) = 0, f_x(3, 3) = f_y(3, 3) = 0, f_x(-3, -3) = f_y(-3, -3) = 0, f_x(1, -1) = f_y(1, -1) = 0, \) and \( f_x(-1, 1) = f_y(-1, 1) = 0, \) those points are critical points.
(b) Compute all the second derivatives \( \partial_{xx}f, \partial_{yy}f, \partial_{xy}f, \partial_{yx}f. \)

\[
\begin{align*}
\partial_{xx}f &= 2y^2 - 10 \\
\partial_{yy}f &= 2x^2 - 10 \\
\partial_{xy}f &= 4xy - 8 \\
\partial_{yx}f &= 4xy - 8 
\end{align*}
\]

(c) Write the characteristic polynomials for each of the 5 critical points in Part (a).

At \((0, 0)\): \( \lambda^2 + 20\lambda + 100 \)
At \((3, 3)\): \( \lambda^2 - 16\lambda + (16 - 28^2) \)
At \((-3, -3)\): \( \lambda^2 - 16\lambda + (16 - 28^2) \)
At \((1, -1)\): \( \lambda^2 + 16\lambda + (-16 - (-12)^2) \)
At \((-1, 1)\): \( \lambda^2 + 16\lambda + (-16 - (-12)^2) \)

(d) Show that \((0, 0)\) is a maximum, and all the rest, \((3, 3), (-3, -3), (1, -1)\) and \((-1, 1)\), are saddle points.

For \((0, 0), \lambda^2 + 20\lambda + 100 = (\lambda + 10)^2 = 0 \implies \lambda_+ = -10, \lambda_- = -10, \) so there is a maximum at this point. We see that \(16 - 28^2 < 0\) and \(-16 - (-12)^2 < 0\), so the rest are saddle points (by the shortcut).

**Problem 2.** Consider the function \( f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24. \)

(a) Show that the only critical point is \((-1, 2)\).

\[
\begin{align*}
f_x(x, y) &= 8x + 8 = 0 \implies x = 1 \\
f_y(x, y) &= 18y - 36 = 0 \implies y = 2 
\end{align*}
\]

(b) Prove that \((-1, 2)\) is a minimum.

\[
\begin{align*}
\partial_{xx}f &= 8 \\
\partial_{yy}f &= 18 \\
\partial_{xy}f &= \partial_{yx}f = 0 
\end{align*}
\]

Since \(8(18) - 0 > 0\) and \(8 > 0\), the point \((-1, 2)\) is a minimum.

**Problem 3.** For each of the following functions, find all critical points and determine whether they are minima, saddles, maxima or cannot decide.

(a) \( f(x, y) = x^3 + 2xy - 6x - 4y^2. \)

\((x_0, y_0) = (-3/2, -3/8)\) - maximum
\((x_1, y_1) = (4/3, 1/3)\) - saddle

(b) \( f(x, y) = x^3 - 3xy^2. \)

\((x_0, y_0) = (0, 0)\) - saddle
(c) $f(x, y) = e^x(x^4 + y^4)$.
$(x_0, y_0) = (-4, 0)$ - can’t decide
$(x_1, y_1) = (0, 0)$ - can’t decide

(d) $f(x, y) = xy - x + y$.
$(x_0, y_0) = (-1, 1)$ - saddle

(e) $f(x, y) = y \cos(x)$.
$(x_n, y_n) = (2\pi n \pm \frac{\pi}{2}, 0)$ for $n \in \mathbb{N}$ - (infinitely many) saddle points

(f) $f(x, y) = x^2 y^2$.
$(x_0, y_0) = (0, 0)$ - can’t decide