MAT 21C: PROBLEM SET 1

DUE TO FRIDAY APR 14 2023

Abstract. This problem set corresponds to the first week of the course MAT-21C Spring 2023. It is due Friday Apr 14 at 9:00am submitted via Gradescope.

Purpose: The goal of this assignment is to acquire the necessary skills to work with sequences (a_n) of real numbers $n \in \mathbb{N}$. These were discussed during the first week of the course and are covered in Section 10.1 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the **Calculus Room** at the Department of Mathematics, and the Office Hours offered by the Professor and the Recitation Instructors. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Textbook: We will use "Thomas' Calculus Early Transcendentals" (15th Edition) by G. B. Thomas, , and J. Hass, Christopher E. Heil, P. Bogacki and M. Weir.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. *In general, you must give a precise reason for why a sequence is divergent or convergent. Similarly, if the sequence is converge, please justify to the best of your ability the claimed limit in your answer. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)*

Problem 1. From "Exercises 10.1" in textbook, in the "Convergence and Divergence" solve 31 through 42 (each item is worth 2 points) and 43 (worth 1 point).

Problem 2. "Exercises 10.1" in textbook in "Convergence and Divergence", solve:

- 101 through 105 (each of these items is worth 2 points)
- 121 through 127 (each of these items is worth 2 points)
- 134 (worth 1 point).

Problem 3. For each of the following statements, justify whether they are true or explain why they are false (e.g. providing a counter-example):

(a) A divergent sequence (a_n) must be unbounded.

(b) A convergent sequence (a_n) must be either decreasing or increasing.

(c) If (a_n) and (b_n) are convergent, then the quotient sequence whose nth term is the quotient $\frac{a_n}{b_n}$ is also convergent.

(d) Let (a_n) and (c_n) be convergent sequences. Then any sequence (b_n) such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$, i.e. (b_n) lies between (a_n) and (c_n), is also convergent.

(e) A sequence (a_n) whose evenly-indexed terms a_{2n} are positive and whose odd-indexed terms a_{2n+1} are negative cannot be convergent.

(*Each item is worth 5 points.*)

Problem 4. For each of the following statements, circle all the correct answers. It might be that no answers are correct, some answers are correct or all answers are correct. If no answers are correct please directly write "No answer is correct.", otherwise circle accordingly.

Just in case: If an arbitrary sequence (a_n) is given, statements such as "(a_n) is increasing" means "(a_n) is increasing for any such given (a_n)". So if an answer is sometimes correct and sometimes incorrect, depending on the choice of possible (a_n), then it must **not** *be circled.*

(1) The sequence $a_n = \frac{3^n}{n!}$ is

(1) increasing. (2) decreasing. (3) convergent. (4) bounded.

(2) If (a_n) is a divergent sequence of positive real numbers and (b_n) is such that $a_n \leq b_n$ for all $n \in \mathbb{N}$, then

(1) (b_n) diverges. (2) (b_n) is unbounded. (3) (b_n) converges.

(3) Let (a_n) be a sequence of *rational* numbers. Then

(1) (a_n) is divergent. (2) If (a_n) converges and limit is rational too. (3) If (a_n) diverges, it is unbounded. (4) (a_n) must be bounded.

(4) The sequence $a_n = n^2 \ln(n)$ growth slower than

(1) $\ln(n)$. (2) n^2. (3) n^3 $n\ln(n)$. (4) $n\ln(\ln(n))$.

(5) The sequence $a_n = (\sqrt{2})^{a_{n-1}}$, $a_1 = \sqrt{2}$, defined recursively, is:

(1) Increasing. (2) Bounded. (3) Convergent. (4) Unbounded.

(6) Suppose (a_n) converges and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. Then the sequence $f(a_n)$

(1) converges. (2) is bounded. (3) is increasing. (4) is decreasing.

(7) Suppose (a_n) diverges, then the sequence $\cos(a_n)$

(1) diverges. (2) is unbounded. (3) might converge or diverge.[1]

(8) Suppose that (a_n) diverges. Then the sequence (b_n) given by $b_n = a_n^2$

(1) diverges. (2) converges. (3) is bounded. (4) is increasing.

(9) Suppose (a_n) is a sequence, $f : \mathbb{R} \longrightarrow \mathbb{R}$ a continuous function and the sequence $f(a_n)$ converges. Then the sequence (a_n)

(1) converges. (2) is bounded. (3) increases. (4) has bounded image $f(a_n)$.

(10) Let (a_n) be defined as $a_n =$ "sum of the first n odd natural numbers". Then the sequence (b_n) given by $b_n := \dfrac{a_n}{n^3 \ln(n)}$, defined for $n \geq 2$,

(1) is bounded (2) converges. (3) is increasing.

(The first 5 items are worth 3 points, the last 5 items are worth 2 points.)

[1]Depending on the given sequence (a_n).