MAT 21C: PRACTICE PROBLEMS LECTURE 2

PROFESSOR CASALS (SECTIONS B01-08)

Abstract. These practice problems correspond to the second lecture of Part II, delivered on May 3rd 2023.

From now onward, the symbol $p \in X$ means p belongs to X.

Problem 1. Consider the plane $\pi = \{3x - 4y + z = 5\}$.

(1) Decide which of the following points belongs to π:

$$(0,0,0), \quad (0,0,5), \quad (1,1,6), \quad (2,-1,4), \quad (0,3,17), \quad (-1,-2,-3),$$

(2) Find two points in π different from those in Part (1).

(3) Find the direction perpendicular to π.

For (1), we substitute the values of x, y and z for every point into the equation $3x - 4y + z = 5$ and see if it is satisfied:

- For $(0,0,0)$: $3 \cdot 0 - 4 \cdot 0 + 1 \cdot 0 \neq 5$, and thus $(0,0,0)$ does *not* belong to π.
- For $(0,0,5)$: $3 \cdot 0 - 4 \cdot 0 + 1 \cdot 5 = 5$, and thus $(0,0,5)$ does belong to π.
- For $(1,1,6)$: $3 \cdot 1 - 4 \cdot 1 + 1 \cdot 6 = 5$, and thus $(1,1,6)$ does belong to π.
- For $(2,-1,4)$: $3 \cdot 2 - 4 \cdot (-1) + 1 \cdot 4 \neq 5$, and $(2,-1,4)$ does *not* belong to π.
- For $(0,3,17)$: $3 \cdot 0 - 4 \cdot 3 + 1 \cdot 17 = 5$, and thus $(0,3,17)$ does belong to π.

So the points that belong to L are $(0,0,5), (1,1,6), (0,3,17)$, while the rest do not.

For (2), we can for instance sample value for x, y and solve for z. For example, choose $x = 1$, $y = 0$ and that $z = 5 - 3x + 4y = 5 - 3 \cdot 1 + 4 \cdot 0 = 2$. So the point $(1,0,2)$ belongs to π. For another one, say we choose $x = 0$, $y = 1$ and then $z = 5 - 3x + 4y = 5 - 3 \cdot 0 + 4 \cdot 1 = 9$. So the point $(0,1,9)$ belongs to π.

For (3), we saw in lecture that the direction perpendicular to a plane of the form $\{ax + by + cz = d\}$ is always (a,b,c). Thus the direction perpendicular to π is $(3,-4,1)$.

Problem 2. Consider the plane π whose perpendicular direction is $(1,2,-5)$ and passes through the point $P = (1,0,1)$. Find an equation for π.

Since the perpendicular direction to π is $(a,b,c) = (1,2,-5)$, the equation for π is of the form $\pi = \{1 \cdot x + 2 \cdot y - 5 \cdot z = d\}$ for some real value d to be found. Since the point P belongs to π, the equation for π must be satisfied for $(1,0,1)$. Therefore we must have $1 \cdot 1 + 2 \cdot 0 - 5 \cdot 1 = d$. This implies $d = -4$ and an equation for π is $\pi = \{1 \cdot x + 2 \cdot y - 5 \cdot z = -4\}$.

Problem 3. Consider three points $P_1 = (1, 0, -1), P_2 = (2, 3, -1)$ and $P_3 = (0, 1, 0)$.

(i) Find the unique plane π which contains P_1, P_2 and P_3.
(ii) Find a different plane π' which also contains both P_1 and P_2, i.e. $P_1, P_2 \in \pi'$, but so that π' does not contain P_3.

For (i), the equation for the plane π must be of the form

$$ax + by + cz = d,$$

for some values of a, b, c, d to be found. Since $P_1, P_2, P_3 \in \pi$, substituting each of their (x, y, z) values must solve the above equation. This yields the following system of three equations for the variables a, b, c, d:

$$\begin{aligned} a \cdot 1 + b \cdot 0 + c \cdot (-1) &= d, \\ a \cdot 2 + b \cdot 3 + c \cdot (-1) &= d, \\ a \cdot 0 + b \cdot 1 + c \cdot 0 &= d. \end{aligned}$$

There are (infinitely many) solutions to this system, we can just pick any of them. For instance, setting $d = 1$ we get $b = 1$ from the last equation. The first two questions then read $a + c = 1$ and $2a + 3 + c = 1$. So we obtain the solution

$$a = -3, b = 1, c = -4, d = 1.$$

Therefore $\pi = \{-3x + y - 4z = 1\}$.

For (ii), let us choose another point P_4 different from P_3 and find the unique plane π' through P_1, P_2 and P_4. For simplicity, we choose $P_4 = (0, 0, 0)$. Then the system of equations for a, b, c, d becomes

$$\begin{aligned} a \cdot 1 + b \cdot 0 + c \cdot (-1) &= d, \\ a \cdot 2 + b \cdot 3 + c \cdot (-1) &= d, \\ a \cdot 0 + b \cdot 0 + c \cdot 0 &= d. \end{aligned}$$

This implies $d = 0$ and we are left with $a - c = 0$ and $2a + 3b - c = 0$. Therefore $a = c$ and $a + 3b = 0$. By choosing $a = 3$ we obtain $c = 3$ and $b = -1$. Hence $\pi' = \{3x - y + 3z = 0\}$.

Problem 4. Consider the plane $\pi = \{5x - 3y + z = -2\}$. Find a different plane π' with the same perpendicular direction as π.

Since π' must have the same perpendicular direction $(a, b, c) = (5, -3, 1)$ as π, an equation for π' must be of the form $\pi' = \{5x - 3y + z = d\}$ for some value of d. It suffices to choose d different from -2 so that π' is different from π. For instance $d = 0$ works, and we can choose $\pi' = \{5x - 3y + z = 0\}$.

Problem 5. Consider the two points $P_1 = (1, 0, -1), P_2 = (2, 3, -1)$. Find the distance between P_1 and P_2.

The formula for the distance $d(P_1, P_2)$ between P_1 and P_2 is

$$d(P_1, P_2) = \sqrt{(1-2)^2 + (0-3)^2 + (-1-(-1))^2} = \sqrt{1 + 9 + 0} = \sqrt{10}.$$