From now onward, the symbol $p \in X$ means $p$ belongs to $X$.

**Problem 1.** Consider the plane $\pi = \{3x - 4y + z = 5\}$.

(1) Decide which of the following points belongs to $\pi$:
- $(0, 0, 0)$, $(0, 0, 5)$, $(1, 1, 6)$, $(2, -1, 4)$, $(0, 3, 17)$, $(-1, -2, -3)$,

(2) Find two points in $\pi$ different from those in Part (1).
(3) Find the direction perpendicular to $\pi$.

For (1), we substitute the values of $x, y$ and $z$ for every point into the equation $3x - 4y + z = 5$ and see if it is satisfied:

- For $(0, 0, 0)$: $3 \cdot 0 - 4 \cdot 0 + 1 \cdot 0 \neq 5$, and thus $(0, 0, 0)$ does not belong to $\pi$.
- For $(0, 0, 5)$: $3 \cdot 0 - 4 \cdot 0 + 1 \cdot 5 = 5$, and thus $(0, 0, 5)$ does belong to $\pi$.
- For $(1, 1, 6)$: $3 \cdot 1 - 4 \cdot 1 + 1 \cdot 6 = 5$, and thus $(1, 1, 6)$ does belong to $\pi$.
- For $(2, -1, 4)$: $3 \cdot 2 - 4 \cdot (-1) + 1 \cdot 4 \neq 5$, and $(2, -1, 4)$ does not belong to $\pi$.
- For $(0, 3, 17)$: $3 \cdot 0 - 4 \cdot 3 + 1 \cdot 17 = 5$, and thus $(0, 3, 17)$ does belong to $\pi$.

So the points that belong to $L$ are $(0, 0, 5), (1, 1, 6), (0, 3, 17)$, while the rest do not.

For (2), we can for instance sample value for $x, y$ and solve for $z$. For example, choose $x = 1, y = 0$ and that $z = 5 - 3x + 4y = 5 - 3 \cdot 1 + 4 \cdot 0 = 2$. So the point $(1, 0, 2)$ belongs to $\pi$. For another one, say we choose $x = 0, y = 1$ and then $z = 5 - 3x + 4y = 5 - 3 \cdot 0 + 4 \cdot 1 = 9$. So the point $(0, 1, 9)$ belongs to $\pi$.

For (3), we saw in lecture that the direction perpendicular to a plane of the form $\{ax + by + cz = d\}$ is always $(a, b, c)$. Thus the direction perpendicular to $\pi$ is $(3, -4, 1)$.

**Problem 2.** Consider the plane $\pi$ whose perpendicular direction is $(1, 2, -5)$ and passes through the point $P = (1, 0, 1)$. Find an equation for $\pi$.

Since the perpendicular direction to $\pi$ is $(a, b, c) = (1, 2, -5)$, the equation for $\pi$ is of the form $\pi = \{1 \cdot x + 2 \cdot y - 5 \cdot z = d\}$ for some real value $d$ to be found. Since the point $P$ belongs to $\pi$, the equation for $\pi$ must be satisfied for $(1, 0, 1)$. Therefore we must have $1 \cdot 1 + 2 \cdot 0 - 5 \cdot 1 = d$. This implies $d = -4$ and an equation for $\pi$ is $\pi = \{1 \cdot x + 2 \cdot y - 5 \cdot z = -4\}$. 
Problem 3. Consider three points $P_1 = (1,0,-1), P_2 = (2,3,-1)$ and $P_3 = (0,1,0)$.

(i) Find the unique plane $\pi$ which contains $P_1, P_2$ and $P_3$.

(ii) Find a different plane $\pi'$ which also contains both $P_1$ and $P_2$, i.e. $P_1, P_2 \in \pi'$, but so that $\pi'$ does not contain $P_3$.

For (i), the equation for the plane $\pi$ must be of the form

$$ax + by + cz = d,$$

for some values of $a,b,c,d$ to be found. Since $P_1, P_2, P_3 \in \pi$, substituting each of their $(x,y,z)$ values must solve the above equation. This yields the following system of three equations for the variables $a,b,c,d$:

$$a \cdot 1 + b \cdot 0 + c \cdot (-1) = d,$$
$$a \cdot 2 + b \cdot 3 + c \cdot (-1) = d,$$
$$a \cdot 0 + b \cdot 1 + c \cdot 0 = d.$$

There are (infinitely many) solutions to this system, we can just pick any of them. For instance, setting $d = 1$ we get $b = 1$ from the last equation. The first two questions then read $a + c = 1$ and $2a + 3 + c = 1$. So we obtain the solution

$$a = -3, b = 1, c = -4, d = 1.$$

Therefore $\pi = \{-3x + y - 4z = 1\}$.

For (ii), let us choose another point $P_4$ different from $P_3$ and find the unique plane $\pi'$ through $P_1, P_2$ and $P_4$. For simplicity, we choose $P_4 = (0,0,0)$. Then the system of equations for $a,b,c,d$ becomes

$$a \cdot 1 + b \cdot 0 + c \cdot (-1) = d,$$
$$a \cdot 2 + b \cdot 3 + c \cdot (-1) = d,$$
$$a \cdot 0 + b \cdot 0 + c \cdot 0 = d.$$

This implies $d = 0$ and we are left with $a - c = 0$ and $2a + 3b - c = 0$. Therefore $a = c$ and $a + 3b = 0$. By choosing $a = 3$ we obtain $c = 3$ and $b = -1$. Hence $\pi' = \{3x - y + 3z = 0\}$.

Problem 4. Consider the plane $\pi = \{5x - 3y + z = -2\}$. Find a different plane $\pi'$ with the same perpendicular direction as $\pi$.

Since $\pi'$ must have the same perpendicular direction $(a,b,c) = (5,-3,1)$ as $\pi$, an equation for $\pi'$ must be of the form $\pi' = \{5x - 3y + z = d\}$ for some value of $d$. It suffices to choose $d$ different from $-2$ so that $\pi'$ is different from $\pi$. For instance $d = 0$ works, and we can choose $\pi' = \{5x - 3y + z = 0\}$.

Problem 5. Consider the two points $P_1 = (1,0,-1), P_2 = (2,3,-1)$. Find the distance between $P_1$ and $P_2$.

The formula for the distance $d(P_1,P_2)$ between $P_1$ and $P_2$ is

$$d(P_1,P_2) = \sqrt{(1-2)^2 + (0-3)^2 + (-1-(-1))^2} = \sqrt{1+9+0} = \sqrt{10}.$$