MAT 21C: PRACTICE PROBLEMS LECTURE 7

PROFESSOR CASALS (SECTIONS B01-08)

ABSTRACT. Practice problems for the seventh lecture of Part II, delivered May 15 2023. Solutions will be posted within 48h of these problems being posted.

Recall the four descriptions of a plane π:

(1) The plane π is given by three points $P, Q, R \in \pi$.
(2) The plane π is given by one point $P \in \pi$ and a normal direction $n = \langle a, b, c \rangle$.
(3) The plane π is given by one point $P \in \pi$ and two vectors u, v inside of π.[1]
(4) The plane π is given by an equation

$$\pi = \{ax + by + cz = d\},$$

where $a, b, c, d \in \mathbb{R}$ are real numbers.

Problem 1. Consider the unique plane π containing the three points $P = (1, 1, 2)$, $Q = (-2, 3, 0)$ and $R = (0, -5, 7)$.

(a) Find two vectors u, v inside of π.
(b) Compute a perpendicular direction to π.
(c) Find an equation for π.

(a) We find the vectors between the points P and Q and between P and R,

$$\overrightarrow{PQ} = \mathbf{u} = \langle -3, 2, -2 \rangle$$
$$\overrightarrow{PR} = \mathbf{v} = \langle -1, -6, 5 \rangle$$

.

(b) We find an orthogonal vector by taking the cross product between $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \times \mathbf{v} = \langle -2, 17, 20 \rangle$$

(c) Using the equation of a plane $\pi = \{\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = d\}$ where $\langle \mathrm{A}, \mathrm{B}, \mathrm{C} \rangle$ is orthogonal to the plane π we get

$$-2x + 17y + 20z = 55$$

.

[1] It is fine if u, v are just two vectors in the direction parallel to π.

Problem 2. Find an equation for the plane through point $(9, 3, -1)$ parallel to the plane $\{x + y + z = 0\}$.

The plane will have orthogonal vector $\mathbf{n} = \langle 1, 1, 1\rangle$. Any plane parallel to $\pi = \{x + y + z = 0\}$ will also be orthogonal to $\mathbf{n}$ so any plane satisfying $\{x + y + z = d\}$ will be parallel to π. Using the point (9,3,-1) we get $d = 11$. Thus the parallel plane has equation $\{x + y + z = 11\}$.

Problem 3. Consider the three planes

$$\pi_1 = \{3x - 5y + 4z = 12\}$$
$$\pi_2 = \{\text{unique plane that contains } (0, 1, 0) \text{ with perpendicular direction } \langle 1, 4, 3\rangle\}$$
$$\pi_3 = \{\text{unique plane that contains } (0, 0, 0) \text{ and vectors } u = \langle 2, 4, 1\rangle, v = \langle 2, -5, 12\rangle\}$$

(a) Show that π_1 intersects π_2 at a line, π_1 intersects π_3 at a line, and π_2 intersects π_3 at a line. (That is, these are not parallel to each other.)

(b) Find the directions of each of these lines.

(a) To show that planes $\{\mathrm{A}x+\mathrm{B}y+\mathrm{C}z=D\}$ and $\{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z=d\}$ intersect at a line we must show that the cross product of $\langle\mathrm{A,B,C}\rangle \times \langle\mathrm{a,b,c}\rangle \neq 0$.

$$\pi_1 = \{x + 4y + 3z = 12\}$$
$$\pi_2 = \{x + 4y + 3z = 4\}$$
$$\pi_3 = \{53x - 22y - 18z = 0\}$$

$$\pi_1 \text{ x } \pi_2 = \langle -31, -5, 17\rangle \neq 0$$
$$\pi_1 \text{ x } \pi_3 = \langle 178, 266, 199\rangle \neq 0$$
$$\pi_2 \text{ x } \pi_3 = \langle -6, 177, -234\rangle \neq 0$$

(b) The direction of each line of intersection is given by the vector calculated by the cross product of the pairs of planes found in part (a).

Problem 4. Consider the two planes

$$\pi_1 = \{3x + 3y + 3z = 12\}$$
$$\pi_2 = \{\text{unique plane that contains } (0,0,0) \text{ with perpendicular direction } \langle 1,1,1\rangle\}$$

(a) Show that π_1 and π_2 are parallel planes and they are different.

(b) Find a plane π_3 different than π_1 and π_2 but is parallel to both of them.

(a) First, we notice that the point (0,0,0) which is in π_1 does not satisfy the equation for π_2 so the planes must be different. We show the planes are parallel by showing the cross product of the planes' normal vectors is zero,

$$\langle 3,3,3\rangle \times \langle 1,1,1\rangle = 0$$

.

(b) Parallel planes have equal variable coefficients but different constant values. Thus, any plane with $\{3x+3y+3z = d\}$ where $d \neq 12$ will be parallel to π_1, such as $\{3x+3y+3z = \text{-}1\}$.

Problem 5. Consider the plane $\pi = \{2x + 9y - z = 3\}$.

(a) Find three distinct points $P, Q, R \in \pi$ that belong to π.

(b) Find two vectors u, v which are parallel to π.

(c) Find a plane π' parallel to π but different from it.

(d) Find a plane π'' which intersects π at a line.

(a) We must find 3 points (x, y, z) such that x, y, and z satisfy the equation for π. Three examples are (2,1,10), (5,-2,-11), and (1,-1,10).

(b) An orthogonal vector to π is $\mathbf{n} = \langle 2, 9, -1\rangle$. Vectors parallel to π will also be orthogonal to $\mathbf{n}$. To find a vector orthogonal to $\mathbf{n}$ we must find a vector whose dot product with $\mathbf{n}$ is zero. Thus, we are looking for a vector $\mathbf{v}$ such that

$$\langle v_1, v_2, v_3\rangle \cdot \langle 2, 9, -1\rangle = 0$$
$$2v_1 + 9v_2 - v_3 = 0$$

Choose an arbitrary v_1 and v_2 such as $v_1 = 3$ and $v_2 = -1$. Solving for v_3 gives $v_3 = -3$, and thus $\mathbf{v} = \langle 3, -1, -3\rangle$. Using the same method we calculate another parallel vector to π is $\mathbf{u} = \langle 6, -2, -6\rangle$.

(c) A parallel plane will have $\{2x+9y-z = d\}$ for $d \neq 3$ such as $\pi' = \{2x+9y-z = 17\}$.

(d) To find a plane that intersects π we must find a plane that is not parallel to π. One such plane is $\pi'' = \{3x + 10y - 2z = 3\}$.

Problem 6. Consider the plane $\pi = \{2x + y - z = 0\}$ and the unique line L through the origin and the point $P = (0, 1, 1)$.

(a) Argue that the point $P \in \pi$ belongs to the plane π.

(b) Justify that the line L lies inside the plane π.

(c) Find a plane π' such that their intersection is the line L.

(a) Plugging in the point $P = (0, 1, 1)$ to the equation for plane π, we get

$$2(0) + (1) - (1) = 0$$

so P satisfies the equation for π showing that P belongs to the plane π.

(b) The line L passes through the points (0,0,0) and (0,1,1) both of which belong to plane π so L must lie in the plane π.

(c) We are looking for a plane $\pi' = \{\mathrm{a}x + \mathrm{b}y + \mathrm{c}z = 0\}$ whose intersection with π is in the direction $\langle 0, 1, 1\rangle$. We use the cross product,

$$\langle 2, 1, -1\rangle \times \langle a, b, c\rangle = \langle 0, 1, 1\rangle$$

. Using the determinant form of the cross product we get the equation

$$\langle c + b, -2c - a, 2b - a\rangle = \langle 0, 1, 1\rangle$$

which produces the system of equations:

$$\begin{aligned} c + b &= 0 \\ -2c - a &= 1 \\ 2b - a &= 1 \end{aligned}$$

Choosing $a = 1$, we solve the system to find $b = 1$ and $c = -1$. Thus $\pi' = \{x + y - z = 0\}$.