MAT 21C: PRACTICE PROBLEMS LECTURE 8

PROFESSOR CASALS (SECTIONS B01-08)

ABSTRACT. Practice problems for the eighth lecture of Part II, delivered May 17 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Consider the unique plane $\pi$ containing the three points $P = (1, 0, 2), Q = (-2, 3, 0)$ and $R = (0, -5, 1)$.

(a) Find the distance from the point $S = (1, 2, -4)$ to $\pi$ using the vector $\vec{PS}$. Recall that the distance from a point $S$ to a plane through some point $X$ with a normal $n$ is given by $d = |\vec{XS} \cdot \frac{n}{|n|}|$.

We will take the cross product of $\vec{PQ}$ and $\vec{PR}$ to find the normal vector. Hence, $n = (-3, 3, -2) \times (-1, -5, -1) = (-13, -1, 18)$ and $|n| = \sqrt{494}$.

Use the vector $\vec{PS} = (0, 2, -6)$.

Thus, $d = |(0, 2, -6) \cdot \left(\frac{-13}{\sqrt{494}}, \frac{-1}{\sqrt{494}}, \frac{18}{\sqrt{494}}\right)| = \left|\frac{-2}{\sqrt{494}} - \frac{108}{\sqrt{494}}\right|$

(b) Find the distance from the point $S = (1, 2, -4)$ to $\pi$ using the vector $\vec{QS}$.

Use the vector $\vec{QS} = (3, -1, -4)$.

Thus, $d = |(3, -1, -4) \cdot \left(\frac{-13}{\sqrt{494}}, \frac{-1}{\sqrt{494}}, \frac{18}{\sqrt{494}}\right)| = \left|\frac{-30}{\sqrt{494}} + \frac{1}{\sqrt{494}} - \frac{72}{\sqrt{494}}\right|$

(c) Find the distance from the point $S = (1, 2, -4)$ to $\pi$ using the vector $\vec{RS}$.

Use the vector $\vec{RS} = (1, 7, -5)$.

Thus, $d = |(1, 7, -5) \cdot \left(\frac{-13}{\sqrt{494}}, \frac{-1}{\sqrt{494}}, \frac{18}{\sqrt{494}}\right)| = \left|\frac{-13}{\sqrt{494}} - \frac{7}{\sqrt{494}} - \frac{90}{\sqrt{494}}\right|$

Problem 2. Consider the three planes

\[ \pi_1 = \{3x - 5y + 4z = 12\} \]
\[ \pi_2 = \{\text{unique plane that contains } (0, 1, 0) \text{ with perpendicular direction } (1, 4, 3)\} \]
\[ \pi_3 = \{\text{unique plane that contains } (0, 0, 0) \text{ and vectors } u = (2, 4, 1), v = (2, -5, 12)\} \]

and the point $S = (-2, 0, 1)$.

(a) Find the distance of $S$ to $\pi_1$.

If a plane is of the form $Ax + By + Cz = D$, then $n = (A, B, C)$. Thus, $n = (3, -5, 4)$ and $|n| = \sqrt{50}$. Take the point $X = (4, 0, 0)$ to get $\vec{XS} = (-6, 0, 1)$.

Thus, $d = |\vec{XS} \cdot \frac{n}{|n|}| = \left|\frac{-18}{\sqrt{50}} + \frac{4}{\sqrt{50}}\right|$

(b) Find the distance of $S$ to $\pi_2$.

Here, $n = (1, 4, 3), |n| = \sqrt{26}$, and $\vec{XS} = (-2, -1, 1)$. Hence, $d = \left|(-2, -1, 1) \cdot \left(\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}\right)\right| = \left|\frac{-2}{\sqrt{26}} - \frac{4}{\sqrt{26}} + \frac{3}{\sqrt{26}}\right|$
(c) Find the distance of $S$ to $\pi_3$.

Here, $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \langle 53, -22, 18 \rangle$, $|\mathbf{n}| = \sqrt{3617}$, and $X\mathbf{n} = \langle -2, 0, 1 \rangle$. Thus,

$$d = \left| \frac{-106}{\sqrt{3617}} + \frac{18}{\sqrt{3617}} \right|$$

**Problem 3.** Find two different points $S_1$ and $S_2$ in space such that both $S_1$ and $S_2$ have distance to the plane $\{x + y + z = 0\}$ equal to 9.

Notice that the plane passes through the point $(0,0,0)$ and has a normal $\mathbf{n} = \langle 1, 1, 1 \rangle$, where $|\mathbf{n}| = \sqrt{3}$. We want to find two possible vectors $X\mathbf{n} = \langle a, b, c \rangle$ such that $\langle a, b, c \rangle \cdot \frac{1}{\sqrt{3}}\langle 1, 1, 1 \rangle = 9 \implies |a+b+c| = 9\sqrt{3}$. We can say that $S_1 = (3\sqrt{3}, 3\sqrt{3}, 3\sqrt{3})$ and $S_2 = (0,0, 9\sqrt{3})$.

**Problem 4.** Find two different planes $\pi_1$ and $\pi_2$ in space such that both $\pi_1$ and $\pi_2$ have distance to the point $S = (1,0,0)$ equal to 23.

Arbitrarily set both planes to have normal $\mathbf{n} = \langle 1, 0, 0 \rangle$. We want to find two points that each respective plane passes through to satisfy the prompt above. Notice that with our choice of normal vectors, the planes $\pi_1$ and $\pi_2$ are parallel with the $yz$-plane, so this problem reduces to a one-dimensional problem. Any planes with $\mathbf{n} = \langle 1, 0, 0 \rangle$ and pass through the point $x = 24$ or $x = -22$ satisfy our conditions.

More precisely, if our planes pass through the point $X = (a, b, c)$, then $d = |\langle 1-a, -b, -c \rangle \cdot (1,0,0)| = |1-a| = 23 \implies 1-a = 23 \text{ or } a-1 = 23$.

**Problem 5.** Consider the two planes

$\pi_1 = \{x - z = 12\}$

$\pi_2 = \{\text{unique plane that contains } (0, 0, 0) \text{ with perpendicular direction } \langle 1, 1, 1 \rangle\}$

(a) Compute the distance from $S = (11, 2, -4)$ to the plane $\pi_1$. The plane $\pi_1$ contains the point $X = (12, 0, 0)$ and has normal line $\mathbf{n} = \langle 1, 0, -1 \rangle$, so $d = |\langle -1, 2, -4 \rangle \cdot \langle \frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}} \rangle| = \left| \frac{-1}{\sqrt{3}} + \frac{4}{\sqrt{3}} \right|$

(b) Compute the distance from $S = (11, 2, -4)$ to the plane $\pi_2$. $d = |\langle 11, 2, -4 \rangle \cdot \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle| = \left| \frac{11}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{4}{\sqrt{3}} \right|$

(c) Compute the distance from $S = (11, 2, -4)$ to the intersection line $\pi_1 \cap \pi_2$.

The distance from some point $S$ to a line that passes through $P$ parallel to $\mathbf{v}$ is $d = \frac{|PS \times \mathbf{v}|}{|\mathbf{v}|}$. Recall that the intersection line is parallel to $n_1 \times n_2$ where $n_1$ and $n_2$ are the normals of $\pi_1$ and $\pi_2$ respectively, so $\mathbf{v} = n_1 \times n_2 = \langle 1,0,-1 \rangle \times \langle 1,1,1 \rangle = \langle 1,-2,1 \rangle$. To obtain the point $P$ we find any $(x,y,z)$ that satisfy $x-z = 12$ and $x+z+y = -12$. Arbitrarily set $z = 0$. Then $P = (12, -12, 0)$. Thus, $d = \frac{|\langle -1,14,-4 \rangle \times \langle 1,-2,1 \rangle|}{|\langle 1,-2,1 \rangle|} = \frac{|\langle 6,-3,-12 \rangle|}{|\langle 1,-2,1 \rangle|} = \frac{3\sqrt{21}}{\sqrt{6}}$

**Problem 6.** Let $L$ be the unique line through the point $P = (1,2,0)$ and direction vector $\mathbf{v} = \langle 0,2,-7 \rangle$. Compute the distance from the point $S = (-3,0,4)$ to the line $L$.

$$d = \frac{|-4,-2,4 \times (0,2,-7)|}{|(0,2,-7)|} = \frac{|6,-28,-8|}{|(0,2,-7)|} = \frac{2\sqrt{221}}{\sqrt{53}}$$
Problem 7. Let $L$ be the unique line through the points $P = (1, 2, 0)$ and $Q = (7, -5, 6)$. Compute the distance from the point $S = (-3, 0, 4)$ to the line $L$.

Here, $v = \overrightarrow{PQ} = (6, -7, 6)$. Thus, $d = \frac{|(-4, -2, 4) \times (6, -7, 6)|}{|(6, -7, 6)|} = \frac{|(16, 48, -40)|}{|(6, -7, 6)|} = \frac{8\sqrt{65}}{\sqrt{121}}$

Problem 8. Decide whether each of the following sentences is true or false.

(a) A point $P$ belongs to a line $L$ if and only if the distance from $P$ to $L$ is zero.
   True. The distance is zero when $\overrightarrow{P_0P} \times v = \vec{0}$. If a point $P$ belongs to the line, then $\overrightarrow{P_0P}$ is parallel to $v$ and $\overrightarrow{P_0P} \times v = \vec{0}$ is guaranteed.

(b) A point $P$ belongs to a plane $\pi$ if and only if the distance from $P$ to $\pi$ is zero.
   True. If a point $P$ belongs to a plane, then $\overrightarrow{P_0P}$ is perpendicular to the plane’s normal vector $\mathbf{n}$. Thus, $\overrightarrow{P_0P} \cdot \mathbf{v} = 0 \implies d = 0$.

(c) Given a point $P$, there exists a unique plane $\pi$ whose distance to $P$ is 1.
   False. Consider the point $P(0, 0, 0)$. The planes $z = 1$ and $x = 1$ are one unit away from the origin, but are two different planes.

(d) Given a point $P$, there are infinitely many lines $L$ whose distance to $P$ is 14.
   True. We can find some plane $\pi$ whose distance to $P$ is also 14, then we generate infinitely many lines within $\pi$ where all the lines pass through the point nearest to $P$. For example, consider the point $(0,0,14)$ and any line in the $xy$-plane that passes through the origin.

(e) If a point $P$ belongs to a plane $\pi_1$ and $L$ is a line of intersection between $\pi_1$ and a different (non-parallel) plane $\pi_2$. Then the distance from $P$ to $L$ is the same as the distance from $P$ to $\pi_2$.
   False. Consider $\pi_1$ as the plane $z = 0$ (the $xy$-plane), $\pi_2$ as the plane $0.001x + z = 0$ (a slight rotation of the $xy$-plane about the y-axis), and the point $P(1000,0,0)$. The distance between $P$ and $\pi_2$ is 1, but the distance between $P$ and $L$ is 1000.