MAT 21C: PRACTICE PROBLEMS LECTURE 9

PROFESSOR CASALS (SECTIONS B01-08)

ABSTRACT. Practice problems for the ninth lecture of Part II, delivered May 19 2023. Solutions will be posted within 48h of these problems being posted.

Problem 1. Find the parametric description of the trajectory of a particle which moves along the unique line through the point $P = (3, -5, 11)$ and with direction vector $\langle 7, 2, -1\rangle$.

The unique line that satisfies this equation is as follows:

$$\begin{aligned} x &= 3 + 7t \\ y &= -5 + 2t \\ z &= 11 - t. \end{aligned}$$

Notice at $t = 0$, we are at point P, and we are moving along the direction vector.

Problem 2. Find the parametric description of the trajectory of a particle which moves along the unique line given by the intersection of the two planes $\pi_1 = \{x+4y-6z = 10\}$ and $\pi_2 = \{11x - 2y - 6z = 3\}$.

To do this, we first need to find the direction of the line, which we can do by taking the cross product of the normal vectors of the two planes. The normal vector of π_1 is $\vec{n_1} = \langle 1, 4, -6\rangle$ and the normal vector is π_2 is $\vec{n_2} = \langle 11, -2, -6\rangle$. Then, the cross product of these two vectors is:

$$\vec{n_1}\times\vec{n_2} = \langle 1, 4, -6\rangle\times\langle 11, -2, -6\rangle = \langle -24+12, -(-6+66), -2-44\rangle = \langle -12, -60, -46\rangle.$$

Then, we need to find a point where the planes intersect. Fix $z = 0$, and then we will find the intersection of

$$\begin{aligned} x + 4y &= 10 \\ 11x - 2y &= 3 \end{aligned}$$

Then, the point of intersection is $\left(\frac{16}{23}, \frac{107}{46}, 0\right)$. We can then find the parametric description like we would find the parametric equation of a line:

$$\begin{aligned} x &= \frac{16}{23} - 12t \\ y &= \frac{107}{46} - 60t \\ z &= -46t. \end{aligned}$$

Problem 3. Suppose that a particle is moving along a line L with its trajectory being given by $r(t) = \langle 3+8t, 5-9t, 1+t\rangle$.

(a) Find the position of the particle at $t = 0$ and $t = 5$.
We can plug in $t = 0$, and we find that

$$r(0) = \langle 3, 5, 1\rangle.$$

If we plug in $t = 5$, we find that

$$r(5) = \langle 43, -40, 6\langle.$$

(b) Find the velocity of the particle at $t = 0$ and $t = 5$.
First, we take the derivative of the position vector, term by term, and we get

$$r'(t) = \langle 8, -9, 1\rangle$$

which is a constant vector. Thus, any t value we plug in will result in the same vector, so

$$r'(0) = r'(5) = \langle 8, -9, 1\rangle.$$

(c) Give a point in the line L and a vector in the direction of the line L.
A point on the line L is a point where $t = 0$, which is

$$r(0) = (3, 5, 1)$$

and the direction vector is given by the velocity vector, which is

$$r'(t) = \langle 8, -9, 1\rangle.$$

Problem 4. Consider a particle in space moving according to the trajectory

$$r(t) = \langle t^2 - 3t, 3 - t, 45t + e^t\rangle.$$

(a) What is the position of the particle at $t = 0$? And at $t = 10$?
If $t = 0$, we plug that into the position vector to get

$$r(0) = \langle 0, 3, 1\rangle$$

and if we plug in $t = 10$ into the position vector, then

$$r(10)\langle 70, -7, 450 + e^{10}\rangle.$$

(b) Compute the velocity vector of the particle at $t = 0$ and at $t = 10$.
First, we can take the derivative which is

$$r'(t) = \langle 2t - 3, -1, 45 + e^t\rangle.$$

Then, this is the velocity vector, and we can plug in different values of t. Plugging in $t = 0$, we get

$$r'(0) = \langle -3, -1, 46\rangle$$

and plug in $t = 10$, we get

$$r'(10) = \langle 17, -1, 45 + e^{10}\rangle.$$

(c) Find the speed of the particle at $t = 0$ and at $t = 10$.
To find the speed, we need to take the length of the velocity vectors. We find that at $t = 0$, the speed is

$$|r'(0)| = \sqrt{3^2 + 1^2 + 46^2} = \sqrt{2126}$$

and at $t = 10$, the speed is

$$|r'(10)|\sqrt{17^2 + 1^2 + (45 + e^{10})^2} = \sqrt{2315 + 90e^{10}e^{20}}.$$

Problem 5. A particle in space moves according to the trajectory

$$r(t) = \langle \cos(t^3), \sin(t^3), 0\rangle.$$

(a) Show that at $t = 0$ and at $t = \sqrt[3]{2\pi}$ the particle passes through the same point.
If we plug $t = 0$ into the position trajectory, we get

$$r(0) = \langle 1, 0, 0\rangle$$

and if we plug $t = (2\pi)^{1/3}$, we get

$$r\left((2\pi)^{1/3}\right) = \langle \cos(2\pi), \sin(2\pi), 0\rangle = \langle 1, 0, 0\rangle.$$

(b) Compute the speed at which is passes at $t = 0$ and at $t = \sqrt[3]{2\pi}$.
First, we can take the derivative to get the velocity vector:

$$r'(t) = \langle -3t^2 \sin(t^3), 3t^2 \cos(t^3), 0\rangle.$$

Then, we can plug in the different t values and find the length of the vector to find the speed. First, we plug in $t = 0$:

$$r'(0) = \langle 0, 0, 0\rangle$$

and we can see the length of the vector is 0. Then, we plug in $t = (2\pi)^{1/3}$, then we find

$$r'\left((2\pi)^{1/3}\right) = \langle 0, 3(2\pi)^{1/3}, 0\rangle.$$

If we take the length of the velocity vector here, we find the length is the speed which is

$$\left|r'\left((2\pi)^{1/3}\right)\right| = (2\pi)^{1/3}.$$

Problem 6. Given a particle moving according to the trajectory

$$r(t) = \langle \cos(t), \sin(t), 5e^{-t} \rangle.$$

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(a) Find the position of the particle at $t = 0$ and $t = \pi$.
To find the position, we have to plug in the t values into the position vector. At $t = 0$, we get

$$r(0) = \langle 1, 0, 5 \rangle$$

and at $t = \pi$, we find that

$$r(\pi) = \langle -1, 0, 5e^{-\pi} \rangle.$$

(b) Find the velocity of the particle at $t = 0$ and $t = 5$.
To find the velocity vector, we first take the derivative of the position vector, and we get

$$r'(t) = \langle -\sin(t), \cos(t), -5e^{-t} \rangle.$$

Then, to find the different velocities at different times, we plug in different t values. We get that at $t = 0$,

$$r'(0) = \langle 0, 1, -5 \rangle$$

and at $t = 5$, we get

$$r'(5) = \langle -\sin(5), \cos(5), -5e^{-5} \rangle.$$

(c) Find the acceleration of the particle at $t = 0$ and $t = 5$.
To find the acceleration vector, we first take the derivative of the velocity vector, and we get

$$r''(t) = \langle -\cos(t), -\sin(t), 5e^{-t} \rangle.$$

Then, to find the acceleration at different times, we plug in different t values. We get that at $t = 0$,

$$r''(0) = \langle -1, 0, 5 \rangle$$

and at $t = 5$,

$$r'(5) = \langle -\cos(5), -\sin(5), 5e^{-5} \rangle.$$

(d) Compute the angle of the velocity and the acceleration at $t = 0$ and $t = 5$.
We can do this by using the dot product formula:

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. We find that for the angle between the velocity vectors, we get:

$$\cos(\theta_v) = \frac{r'(0) \cdot r'(5)}{|r'(0)||r'(5)|} = \frac{\cos(5) + 25e^{-5}}{\sqrt{25+1}\sqrt{\sin^2(5) + \cos^2(5) + 25e^{-10}}} = \frac{\cos(5) + 25e^{-5}}{\sqrt{26}\sqrt{1 + 25e^{-10}}}.$$

For the angle between the acceleration vectors, we get

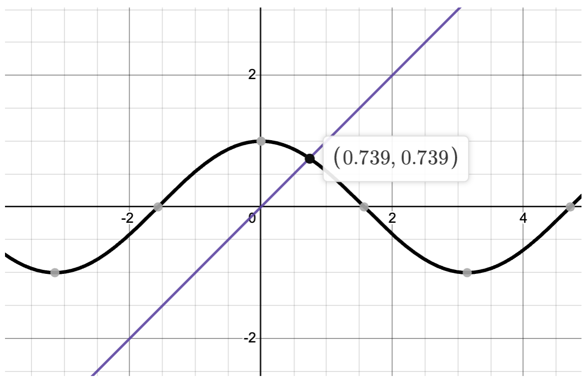
$$\cos(\theta_a) = \frac{r''(0) \cdot r''(5)}{|r''(0)||r''(5)|} = \frac{\cos(5) + 25e^{-5}}{\sqrt{25+1}\sqrt{\sin^2(5) + \cos^2(5) + 25e^{-10}}} = \frac{\cos(5) + 25e^{-5}}{\sqrt{26}\sqrt{1 + 25e^{-10}}}.$$

We notice that
$$\cos(\theta_v) = \cos(\theta_a)$$
which means that
$$\theta_v = \theta_a,$$
so the angle between the two velocity vectors and acceleration vectors is the same.

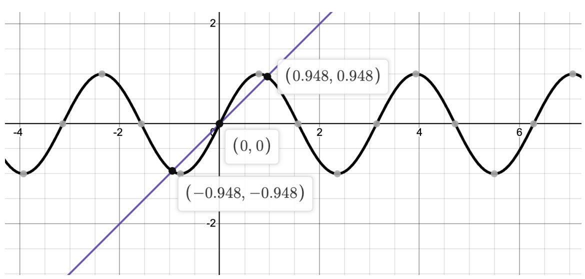
Problem 7. Two particles move in space according to the trajectories
$$r_1(t) = \langle \cos(t), \sin(t), t \rangle, \qquad r_1(t) = \langle t, 2t, t \rangle.$$

Show that the two particles will never collide.

If we compare $\cos(t)$ and t, we see they should intersect at one place: If we compare



$\sin(t)$ and $2t$, they intersect three times: However, none of these t values are the same,



so there is no common value of t for all three common equations. Thus, the space curves will never intersect.

Problem 8. Two particles move in space according to the trajectories
$$r_1(t) = \langle \cos(t), \sin(t), t \rangle, \qquad r_1(t) = \langle 1, 0, t \rangle.$$

Show that the two particles will collide infinitely many times and find all such times of collision.

Notice that $\cos(t) = 1$ if $t = 2n\pi$, where n is an integer. When $t = 2n\pi$, then $\sin(t)$ is 0 as well. Thus, these two particles will collide infinitely many times at
$$t = 2n\pi$$
where n is an integer.

Problem 9. Suppose that a particle travels along a line L through the point $P = (p_1, p_2, p_3)$ and vector direction $v = \langle v_1, v_2, v_3 \rangle$ via the parametric trajectory

$$r(t) = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle.$$

(a) Show that the velocity vector of $r(t)$ is exactly the direction vector v of L.
Notice the direction vector of $r(t)$ is

$$v = \langle v_1, v_2, v_3 \rangle$$

and the derivative of $r(t)$ is

$$r'(t) = \langle v_1, v_2, v_3 \rangle.$$

These are the same vector.

(b) Show that the acceleration vector of $r(t)$ is always zero.
If we take the derivative of the velocity vector $r'(t)$, we get the $\vec{0}$ vector:

$$r''(t) = \langle 0, 0, 0 \rangle.$$

Problem 10. Find a parametric description of the curve given by $r(t) = \langle 1, 3t + 2, t \rangle$ where the particle moves exactly in the same trajectory but at twice the speed.

To double the speed of the particle, first let's find the velocity vector:

$$r'(t) = \langle 0, 3, 1 \rangle.$$

To compute the speed, we take the length of this vector and find that

$$|r'(t)| = \sqrt{0^2 + 3^2 + 1^2} = \sqrt{10}.$$

To double this speed, let's try doubling the velocity vector, and putting that velocity vector (or the direction vector, as we showed in the last problem) back into the original vector with the same starting point. So the new velocity vector is

$$R'(t) = \langle 0, 6, 2 \rangle$$

and the new position vector is

$$R(t) = \langle 1, 6t + 2, 2t \rangle.$$

If we take the length of the new velocity vector, we get

$$|R'(t)| = \sqrt{0^2 + 6^2 + 2^2} = \sqrt{40} = 2\sqrt{10} = 2|r'(t)|.$$

We can see this is double the speed of the original particle's speed, but it still moves along the same trajectory. In fact, at $t = 0$, we see that

$$r'(0) = R'(0).$$