University of California Davis **Name (Print):** ____________________

Calculus MAT 21C **Student ID (Print):** ____________________

Midterm Examination **April 28 2023**

Time Limit: 50 Minutes

This examination document contains 9 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Consider the sequence $a_n = \frac{2^n}{n!}$, where n denotes a natural number $n \geq 1$.

   (a) (5 points) Write the first 4 terms a_1, a_2, a_3, a_4 of this sequence.

   We have

$$a_1 = \frac{2^1}{1!}, \qquad a_2 = \frac{2^2}{2!}, \qquad a_3 = \frac{2^3}{3!}, \qquad a_4 = \frac{2^4}{4!}.$$

   Thus the first four terms are $2, 2, 4/3$ and $2/3$.

   (b) (5 points) Justify that $a_{n+1} \leq a_n$ for all n. (So the sequence (a_n) is decreasing.)

   In our case, the inequality $a_{n+1} \leq a_n$ is

$$\frac{2^{n+1}}{(n+1)!} \leq \frac{2^n}{n!}.$$

   This is true because

$$\frac{2^{n+1}}{(n+1)!} \leq \frac{2^n}{n!} \iff \frac{2^{n+1}}{2^n} \leq \frac{(n+1)!}{n!} \iff 2 \leq n+1 \iff 1 \leq n.$$

(c) (5 points) Argue that $a_n \geq 0$ for all $n \geq 1$. (So the sequence (a_n) is bounded below.)

The numerator 2^n is positive and so is the denominator $n!$. Therefore the quotient $a_n = \frac{2^n}{n!}$ is also positive.

(d) (5 points) Show that (a_n) is convergent.

We apply the Monotone Convergence Theorem. Part (b) shows that (a_n) is decreasing and Part (c) shows that a_n is bounded below. The Monotone Convergence Theorem then concludes that a_n converges.

Alternatively, in the hierarchy of functions $n!$ grows orders of magnitude faster than 2^n. Therefore the limit of 2^n over $n!$ exists and it is zero.

(e) (5 points) Explain why the limit $\lim_{n\to\infty} a_n = 0$ is 0.

As said above, via the hierarchy of growth of functions $n!$ grows orders of magnitude faster than 2^n and thus $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

Alternatively, we can use comparison to $\frac{1}{n}$, since $\frac{2^n}{n!} \leq \frac{1}{n}$ as explained in lecture. By the squeezing theorem

$$0 \leq \lim_{n\to\infty} \frac{2^n}{n!} \leq \lim_{n\to\infty} \frac{1}{n} = 0,$$

and thus $\lim_{n\to\infty} \frac{2^n}{n!} = 0$.

2. (25 points) Solve the two parts below

   (a) (20 points) For each of the series below, determine whether the series converges or diverges. You *must* justify your answer in detail. If you are applying a certain test, **state the name of the test clearly, the steps implementing the test and its outcome**. If a sequence converges, you do *not* need to find the limit.

$$1. \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n+1}}, \qquad 2. \sum_{n=1}^{\infty} \frac{2n^6+15n^2+1}{n^6-9n^2-10}, \qquad 3. \sum_{n=1}^{\infty} \frac{3^n}{n!}, \qquad 4. \sum_{n=1}^{\infty} \frac{\ln(n))^n}{n^n}.$$

1. **Converges**. The alternating series test shows convergence. The test is applied to $b_n = \frac{1}{\sqrt[3]{n+1}}$, which is decreasing, converging to 0 and positive.

2. **Diverges**. The terms $a_n = \frac{2n^6+15n^2+1}{n^6-9n^2-10}$ being added do not converge to 0. (They converge to 2.) Therefore the series diverges.

3. **Converges**. The ratio test shows convergence (so does the root test). The test is applied to $a_n = \frac{3^n}{n!}$ and gives ratio

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty} \frac{3}{n+1} = 0,$$

which is less than 1 and thus the series converges.

3. **Converges**. The root test shows convergence. The test is applied to the terms $a_n = \dfrac{\ln(n))^n}{n^n}$ and the limit is

$$\lim_{n\to\infty} \left| \sqrt[n]{\frac{\ln(n))^n}{n^n}} \right| = \lim_{n\to\infty} \frac{\ln(n)}{n} = 0.$$

This limit is less than 1 and thus the series converges.

(b) (5 points) Explain why the following series converges if and only if the positive real value α satisfies $\alpha > 2$:

$$\sum_{n=1}^{\infty} \frac{n}{n^{\alpha} + 14}.$$

The comparison test shows that this series is comparable to the p-series

$$\sum_{n=1}^{\infty} \frac{n}{n^{\alpha}} = \sum_{n=1}^{\infty} \frac{1}{\frac{n^{\alpha}}{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-1}} = \sum_{n=1}^{\infty} \frac{1}{n^{p}}, \quad \text{where we set } p = \alpha - 1.$$

This is a p-series with $p = \alpha - 1$. Hence our series converges if and only if the p-series with $p = \alpha - 1$ converges. The p-series test (or the integral test, or lecture) shows that a p-series converges if and only if $p > 1$. Thus $p = \alpha - 1$ implies $\alpha > 2$.

3. (25 points) Consider the function $f(x) = \cos(x)$.

(a) (10 points) Compute the five values $f(0)$, $f'(0)$ and $f''(0)$, $f'''(0)$ and $f''''(0)$, i.e. of $\cos(0)$ and the first fourth derivatives of $\cos(x)$ at 0.

We have $\cos(0) = 1$. For the derivatives we have

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f'''(x) = \sin(x), \quad f''''(x) = \cos(x).$$

Therefore $f(0) = 1$, $f'(0) = 0$ and $f''(0) = -1$, $f'''(0) = 0$ and $f''''(0) = 1$.

(b) (10 points) Argue that the Taylor expansion of $\cos(x)$ of order 4 at $a = 0$ is

$$\cos(x) \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!}.$$

The nth term in this Taylor expansion has coefficient $a_n = \frac{f^{(n)}(0)}{n!}$. By Part (a) we have $a_0 = 1, a_1 = 0, a_2 = -1/2! = -1/2, a_3 = 0/3!$ and $a_4 = 1/4!$. Hence the degree 4 part of the Taylor expansion is

$$\cos(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = 1 - \frac{x^2}{2} + \frac{x^4}{4!}.$$

Alternatively, one can use the full Taylor expansion of $\cos(x)$ seen in class or as explained in the textbook and truncate at order 4 to get the expression above.

(c) (5 points) Explain why the Taylor expansion of $\cos(x)$ of order 4 at $a = 0$ approximates the value $\cos(0.1)$ with an error less than 10^{-5}.

Hint: We saw in lecture that the error $R_4(x)$ of the Taylor expansion of $\cos(x)$ of order 4 centered at $a = 0$ is bounded above by $\frac{1}{5!}x^5$.

By the hint, $R_4(x)$ is bounded above by $\frac{1}{5!}x^5$. Therefore if we use the Taylor expansion at order 4 and plug $x = 0.1$ to approximate $\cos(0.1)$ the error will be bounded by $\frac{1}{5!}x^5 = \frac{1}{5!}(0.1)^5$. Since $\frac{1}{5!}(0.1)^5$ is less than 10^{-5}, this justifies the claim.

4. (25 points) For each of the five sentences below, circle the correct answer. There is a unique correct answer per item. (You do *not* need to justify your answer.)

(a) (6 points) The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for p positive real number

(1) converges if $p \geq 1$. (2) converges if $p > 1$. (3) converges for all p.

A p-series converges if and only if $p > 1$. This was done in lecture, it is in the textbook or you can directly apply the integral test. The correct answer is (2).

(b) (6 points) The value of the infinite series $\sum_{n=0}^{\infty} \frac{1}{6^n}$ is

(1) 1/6 (2) 6/5 (3) ∞

This is a geometric series with $r = 1/6$. Therefore it converges and the limit is

$$\frac{1}{1-\frac{1}{6}} = \frac{6}{5}.$$

The correct answer is (2).

(Note also that (1) cannot be right because all terms are positive and the first two terms are 1 and 1/6. So the sum is at least $1 + 1/6$.)

(c) (6 points) The radius of convergence of the power series $\sum_{n=0}^{\infty} x^n$ is

(1) 0 (2) 1/2 (3) 1

This is a geometric series, thus its radius of convergence is $R = 1$. The correct answer is (3).

(d) (4 points) The Taylor series of the polynomial $f(x) = 1 - 3x + x^2$ at $a = 0$ is

(1) $1 - 3x + x^2$ (2) $1 - 3x + x^2 + x^3 + x^4$ (3) $1 - 3x + x^2 - 3x^3 + x^4$

The Taylor series of a polynomial is itself (because the best polynomial approximating a function which is a polynomial is itself). This was discussed in lecture and see also textbook. The correct answer is $1 - 3x + x^2$, that is, (1).

(e) (3 points) The Taylor expansion of x^2e^{-x} at $a = 0$ of order 3 is

(1) $1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}$ (2) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ (3) $x^2 - x^3$

The Taylor expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

Thus the Taylor expansion of e^{-x} is

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots.$$

So the Taylor expansion of x^2e^{-x} is

$$x^2 \cdot (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots) = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \frac{x^6}{4!} + \dots$$

The truncation at order 3 is thus $x^2 - x^3$. The correct answer is (3).

Alternatively, it is clear that x^2e^{-x} evaluated at $a = 0$ is 0, so the constant term in the Taylor series must be 0, that discards (1) and (2) directly, thus the correct answer must be (3).