University of California Davis **Name (Print):** ____________________

Calculus MAT 21C **Student ID (Print):** ____________________

Practice Final Examination II **June 12 2023**

Time Limit: 120 Minutes

This examination document contains 12 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The maximum grade for this exam is **100 points**. Any grade surpassing that mark will still count for 100 points, which is the top grade. You can obtain 100 points in any combination of the problems below, the **extra 20 bonus points** can only help you. You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by no calculations, explanations, or algebraic work will receive less credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. (20 points) Consider a differentiable function $f(x)$ whose Taylor series at $a = 0$ is

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

(a) (4 points) Find the following values $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ of $f(x)$ and its first three derivatives at 0.

Since

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \ldots,$$

then

$$f'(x) = 1 - x^2 + x^4 - x^6 + x^8 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n},$$

$$f''(x) = 0 - 2x + 4x^3 - 6x^5 + 8x^7 + \ldots = \sum_{n=1}^{\infty} (-1)^n 2n x^{2n-1},$$

$$f'''(x) = 0 - 2 + 12x^2 - 30x^4 + 56x^6 + \ldots = \sum_{n=1}^{\infty} (-1)^n 2n(2n-1) x^{2n-2},$$

and these give

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -2.$$

(b) (4 points) Show that the radius of convergence of this Taylor series if $R = 1$.

Using ratio test, and denote

$$a_n = (-1)^n \frac{x^{2n+1}}{2n+1}$$

then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \Longrightarrow \lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+3} \frac{2n+1}{x^{2n+1}} \right| < 1 \Longrightarrow |x|^2 < 1 \Longrightarrow |x| < 1.$$

Hence, we have the radius of convergence $R = 1$.

(c) (4 points) Discuss whether the Taylor series converges at each of the two boundary points $x = 1$ and $x = -1$. (Analyze these points independently.)

When $x = 1$, the above Taylor series equals to

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}.$$

Denote $u_n = 1/(2n+1)$, then u_n's is positive for $n \geq 0$, u_n's are eventually decreasing, and $u_n \to 0$ as $n \to \infty$. Thus, by alternating series test, this series converges.

When $x = -1$, the above Taylor series equals to

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}.$$

Using the same method, the alternating series test, we get this series converges.

(d) (4 points) Consider the following series

$$\sum_{n=0}^{\infty} \frac{3^{2n+1}}{2n+1}.$$

Decide whether the series is convergent or divergent.
(If convergent, you do *not* need to find its limit.)
Using Ratio test, and set $a_n = \frac{3^{2n+1}}{2n+1}$, then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{3^{2n+3}}{2n+3} \frac{2n+1}{3^{2n+1}} \right| = 9 > 1.$$

Hence, by Ratio test, this series diverges.

(e) (4 points) The error of the Taylor approximation of $f(x)$ centered at $a = 0$ of order 8 in the interval $x \in [0, 0.1]$ is bounded by 10^{-8}. Using the Taylor approximation of order 7, find the numerical value of

$$f(0.1) = \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{2n+1}}{2n+1}$$

with an accuracy of 8 decimal digits, i.e. at least 8 decimal digits correct.

The Taylor polynomial of order 7 is

$$P_7(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}.$$

Setting $x = 0.1$, we have

$$f(0.1) \approx P_7(0.1) = 0.1 - \frac{0.1^3}{3} + \frac{0.1^5}{5} - \frac{0.1^7}{7}.$$

2. (20 points) Consider the two planes

$$\pi_1 = \{3x - 5y + z = 8\}, \qquad \pi_2 = \{x + y - z = 1\}.$$

(a) (5 points) Consider the point $S = (0, 1, 0)$. Compute the distance from S to the plane π_1 and the distance from S to the plane π_2.

Setting $x = 0, y = 0$, then $P_1 = (0, 0, 8) \in \pi_1$, $P_2 = (0, 0, -1) \in \pi_2$. The distance from S to π_1 is

$$d_1 = \left| \overrightarrow{P_1S} \cdot \frac{\langle 3, -5, 1\rangle}{|\langle 3, -5, 1\rangle|} \right| = \left| \langle 0, 1, -8\rangle \cdot \frac{\langle 3, -5, 1\rangle}{|\langle 3, -5, 1\rangle|} \right| = \frac{13}{\sqrt{35}},$$

and the distance from S to π_2 is

$$d_2 = \left| \overrightarrow{P_2S} \cdot \frac{\langle 1, 1, -1\rangle}{|\langle 1, 1, -1\rangle|} \right| = \left| \langle 0, 1, 1\rangle \cdot \frac{\langle 1, 1, -1\rangle}{|\langle 1, 1, -1\rangle|} \right| = 0.$$

(b) (5 points) Find a vector $\vec{v}$ in the direction of the line of intersection of π_1 and π_2.

The direction vector of the line of intersection is

$$\vec{v} = \langle 3, -5, 1\rangle \times \langle 1, 1, -1\rangle = \langle 4, 4, 8\rangle$$

(c) (5 points) Calculate the distance from S to the line of intersection of π_1 and π_2.

Setting $x = 0$ for the two plane functions gives

$$-5y + z = 8, \; y - z = 1.$$

Solving these two functions gives

$$y = -\frac{9}{4}, z = -\frac{13}{4},$$

so that we have the point $P = (0, -9/4, -13/4)$ on the line of intersection of the two planes. Then the distance from S to the line of intersection of π_1 and π_2 is

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} = \frac{|\langle 0, 13/4, 13/4\rangle \times \langle 4, 4, 8\rangle|}{|\langle 4, 4, 8\rangle|} = \frac{|\langle 13, 13, -13\rangle|}{|\langle 4, 4, 8\rangle|} = \frac{\sqrt{507}}{\sqrt{96}}$$

(d) (5 points) Consider the ball $\{x^2 + (y-1)^2 + z^2 \leq 0.01\}$. What is the intersection of this ball with π_1? What is the intersection of this ball with π_2?

$$\pi_1 = \{3x - 5y + z = 8\}, \qquad \pi_2 = \{x + y - z = 1\}.$$

Intersection with π_1 is empty set, and the intersection with π_2 is disk, since the center of the ball, (0,1,0), is in π_2.

3. (20 points) Consider the two vectors $u = \langle 5, -4, 2 \rangle$ and $v = \langle 1, 3, 1 \rangle$.

(a) (4 points) Show that $w = \langle 6, -2, 0 \rangle$ is perpendicular to v. Is it perpendicular to u?

Note that two vectors are perpendicular if and only if the dot product of the two vectors equals to 0. Since

$$w \cdot v = \langle 6, -2, 0 \rangle \cdot \langle 1, 3, 1 \rangle = 0,$$

and

$$w \cdot u = \langle 6, -2, 0 \rangle \cdot \langle 5, -4, 2 \rangle = 38,$$

this gives w is perpendicular to v, and w is not perpendicular to u.

(b) (4 points) Justify that $3 \cos \theta = -\sqrt{5/11}$, where θ is the angle between u and v.

Using the dot product formula we have

$$u \cdot v = |u||v| \cos \theta \Longrightarrow -5 = \sqrt{45}\sqrt{11} \cos \theta \Longrightarrow \cos \theta = \frac{-5}{\sqrt{45}\sqrt{11}}.$$

So that

$$3 \cos \theta = -\frac{15}{\sqrt{45}\sqrt{11}} = -\sqrt{\frac{5}{11}}.$$

(c) (4 points) Find the area of the parallelogram spanned by u and v.

The area equals to

$$Area = |u \times v| = |\langle 5, -4, 2\rangle \times \langle 1, 3, 1\rangle| = \langle -10, -3, 19\rangle = \sqrt{470}.$$

(d) (4 points) A particle moves with trajectory

$$r(t) = \langle \cos(t), \sin(t), t\rangle.$$

Find the velocity vector $\dot{r}(t)$ and its speed at $t = 0$.

The velocity is the derivative of $r(t)$, which equals

$$\dot{r}(t) = \langle -\sin(t), \cos(t), 1\rangle.$$

Speed is the length of the velocity vector, which equals

$$|\dot{r}(t)| = \sqrt{\sin^2(t) + \cos^2(t) + 1} = \sqrt{2}.$$

Hence, $\dot{r}(0) = \langle 0, 1, 1\rangle$, and $|\dot{r}(0)| = \sqrt{2}$.

(e) (4 points) Argue that the velocity vector $\dot{r}(t)$ above is never parallel to $\langle 1,3,1\rangle$.

Two nonzero vectors $\mathbf{u}, \mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. Since

$$\dot{r}(t) = \langle -\sin(t), \cos(t), 1\rangle \neq \langle 0,0,0\rangle,$$

and

$$\dot{r}(t)\times\langle 1,3,1\rangle = \langle -\sin(t), \cos(t), 1\rangle\times\langle 1,3,1\rangle = \langle \cos(t)-3, \sin(t)+1, -3\sin(t)-\cos(t)\rangle.$$

Since $-1 \leq \cos(t) \leq 1$, then

$$-4 \leq \cos(t) - 3 \leq -2,$$

and this implies the first coordinate never equals to 0. Hence, $\dot{r}(t)$ never parallel to $\langle 1,3,1\rangle$.

4. (20 points) Consider the function

$$f(x,y) = (x^2-1)^2 + (x^2y - x - 1)^2.$$

(a) (5 points) Compute the partial derivatives $\partial_x f$ and $\partial_y f$.

$$\partial_x f = 2(x^2-1)\cdot 2x + 2(x^2y - x - 1)\cdot(2xy-1)$$

$$\partial_y f = 0 + 2(x^2y - x - 1)\cdot x^2$$

(b) (5 points) Show that the points $(x,y) = (-1,0)$ and $(x,y) = (1,2)$ are both critical points of $f(x,y)$. (You do *not* need to argue that these are all.)

Substitute $(-1,0)$ and $(1,2)$ into the partial derivative functions, we get

$$\partial_x f(-1,0) = 2((-1)^2-1)\cdot 2(-1) + 2((-1)^2\cdot 0 - (-1) - 1)\cdot(2(-1)\cdot 2 - 1) = 0$$

$$\partial_y f(-1,0) = 2((-1)^2\cdot 0 - (-1) - 1)\cdot(-1)^2 = 0,$$

and

$$\partial_x f(1,2) = 2(1^2-1)\cdot 2 + 2(1^2\cdot 2 - 1 - 1)\cdot(2\cdot 1\cdot 2 - 1) = 0$$

$$\partial_y f(1,2) = 2(1^2\cdot 2 - 1 - 1)\cdot 1^2 = 0.$$

Hence, this proves $(-1,0),(1,2)$ are critical points.

(c) (5 points) Compute the second partial derivatives $\partial_{xx} f$, $\partial_{yy} f$, $\partial_{xy} f$, and $\partial_{yx} f$.

Given the partial derivatives from previous part,

$$\partial_x f = 2(x^2-1)\cdot 2x + 2(x^2y-x-1)\cdot(2xy-1) = 4x^3y^2+4x^3-6x^2y-4xy-2x+2$$

$$\partial_y f = 0 + 2(x^2y-x-1)\cdot x^2 = 2x^4y-2x^3-2x^2,$$

we get

$$\partial_{xx} f = 12x^2y^2+12x^2-12xy-4y-2,$$
$$\partial_{yy} f = 2x^4$$
$$\partial_{xy} f = 8x^3y-6x^2-4x$$
$$\partial_{yx} f = 8x^3y-6x^2-4x.$$

(d) (5 points) Classify each of the two critical points $(-1,0)$ and $(1,2)$ into a minimum, a saddle, a maximum or cannot decide.

Substitute the point $(-1,0)$ into the second order partial derivatives, this gives

$$\partial_{xx} f(-1,0) = 10, \partial_{yy} f(-1,0) = 2, \partial_{xy} f(-1,0) = -2, \partial_{yx} f(-1,0) = -2.$$

Since

$$\partial_{xx} f(-1,0)\cdot\partial_{yy} f(-1,0) - \partial_{xy} f(-1,0)\cdot\partial_{yx} f(-1,0) = 24 > 0,$$
$$\partial_{xx} f(-1,0) = 10 > 0,$$

then $(-1,0)$ is a local min.

Substitute $(1,2)$ into second order partial derivatives gives

$$\partial_{xx} f(1,2) = 26, \partial_{yy} f(1,2) = 2, \partial_{xy} f(1,2) = 6, \partial_{yx} f(1,2) = 6.$$

Since

$$\partial_{xx} f(1,2)\cdot\partial_{yy} f(1,2) - \partial_{xy} f(1,2)\cdot\partial_{yx} f(1,2) = 16 > 0,$$
$$\partial_{xx} f(1,2) = 26 > 0,$$

then $(1,2)$ is a local min.

5. (20 points) For each of the five sentences below, circle the correct answer. There is a unique correct answer per item. (You do *not* need to justify your answer.)

   (a) (4 points) The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^\alpha}$ converges

   (1) if $\alpha > 1$. (2) if $\alpha \geq 1$. (3) if $\alpha \leq 1$. (4) converges for all $\alpha > 0$.

   Using alternating series test, as long as the term $1/n^\alpha$ is decreasing to 0, the series converges. Hence, this implies $\alpha > 0$.

   (b) (4 points) The value of the infinite series $\sum_{n=0}^{\infty} \frac{7^n}{5^n}$ is

   (1) $1/6$ (2) $1/4$ (3) $5/2$ (4) $-5/2$ (5) $5/7$ (6) ∞

   This is a geometric series with ratio $7/5 > 1$. Hence, series diverges.

   (c) (4 points) The cross product of $u = \langle 1, -2, -5\rangle$ and $v = \langle 3, -6, -15\rangle$ is

   (1) $\langle 0, 0, 0\rangle$ (2) $\langle 0, 1, 0\rangle$ (3) $\langle 0, -1, 0\rangle$ (4) $\langle 1, 0, 0\rangle$ (5) $\langle -1, 0, 0\rangle$

   Since $3u = v$, this implies u, v are parallel. Hence, the cross product equals zero vector.

   (d) (4 points) The length of $\vec{PQ}$, where $P = (-2, 5, 0)$ and $Q = (3, 4, -3)$, is

   (1) $\sqrt{11}$ (2) $\sqrt{27}$ (3) $\sqrt{35}$ (4) $\sqrt{54}$ (5) 0

   $\vec{PQ} = \langle 3-(-2), 4-5, -3-0\rangle = \langle 5, -1, -3\rangle$, so that the length equals $\sqrt{35}$.

   (e) (4 points) The Taylor series of the function $f(x) = (x+3)^2$ centered at 0 is

   (1) $9+6x+x^2$ (2) $6x+x^2$ (3) $9+6x$ (4) $9-6x$ (5) $9-6x+x^2$ (6) $9+x^2+x^3$

   Since $f(x) = (x+3)^2 = x^2+6x+9$, which is a polynomial. Hence, the Taylor series equals to x^2+6x+9.