University of California Davis **Name (Print):** ______________
Calculus MAT 21C **Student ID (Print):** ______________

Practice Final Examination **June 12 2023**
Time Limit: 120 Minutes

This examination document contains 13 pages, including this cover page, and 5 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The maximum grade for this exam is **100 points**. Any grade surpassing that mark will still count for 100 points, which is the top grade. You can obtain 100 points in any combination of the problems below, the **extra 20 bonus points** can only help you. You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by no calculations, explanations, or algebraic work will receive less credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

Do not write in the table to the right.

1. (20 points) Consider the function $f(x) = \ln(1 + x^2)\cos(4x)$.

   (a) (4 points) Compute the Taylor series of $\cos(4x)$ centered at $a = 0$.

   We use the known expansion for cos (x) then replace x with $4x$.

$$\begin{aligned}
\cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\cos(4x) &= 1 - \frac{(4x)^2}{2} + \frac{(4x)^4}{4!} - \frac{(4x)^6}{6!} + \ldots \\
&= 1 - \frac{16x^2}{2} + \frac{256x^4}{4!} - \frac{4096x^6}{6!} + \ldots \\
&= 1 - 8x^2 + \frac{32x^4}{3} - \frac{256x^6}{45} + \ldots
\end{aligned}$$

   (b) (4 points) Justify why the Taylor series of $\ln(1 + x^2)$ centered at $a = 0$ is

$$\ln(1 + x^2) \approx x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots = \sum_{n \geq 1} (-1)^{n+1} \frac{x^{2n}}{n}.$$

   Using the first four derivatives of $\ln(1 + x)$,

$$\begin{aligned}
f(x)|_{x=0} &= \ln(1) = 0 \\
f'(x)|_{x=0} &= \frac{1}{1+x}|_{x=0} = 1 \\
f''(x)|_{x=0} &= \frac{-1}{(1+x)^2}|_{x=0} = -1 \\
f'''(x)|_{x=0} &= \frac{2}{(1+x)^3}|_{x=0} = 2 \\
f^{(4)}(x)|_{x=0} &= \frac{-6}{(1+x)^4}|_{x=0} = -6
\end{aligned}$$

   and the formula for the Taylor expansion, $f(x) = \sum \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$, we get

$$\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

   Then substituting x^2 for x we get,

$$\ln(1 + x^2) \approx x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \ldots$$

(c) (4 points) Find the radius of converge of the Taylor series in Part (b).

The series will converge for $L < 1$ where L is the limit of the Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\cdot x^{2(n+1)}}{n+1}\cdot\frac{n}{(-1)^{n+1}\cdot x^{2n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^{2n+2}\cdot n}{x^{2n}(n+1)}\right| \\
&= \lim_{n\to\infty}\left|x^2\cdot\frac{n}{n+1}\right| \\
&= |x^2|\cdot\lim_{n\to\infty}\left|\frac{n}{n+1}\right| \\
&= |x^2|
\end{aligned}$$

Thus, $L = |x^2| < 1 \Rightarrow |x| < 1$. Thus, R=1.

(d) (4 points) Argue that the Taylor approximation of order 7 of $f(x)$ at $a = 0$ is

$$f(x) \approx x^2 - \frac{17x^4}{2} + 15x^6.$$

To find the Taylor approximation of $f(x) = \ln(1+x)\cos(4x)$, we multiply the Taylor approximations of each function and keep only terms of order 7 and lower.

$$\begin{aligned}
\ln(1+x^2)\cos(4x) &\approx \left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + ...\right)\left(1 - 8x^2 + \frac{32x^4}{3} - \frac{256x^6}{45} + ...\right) \\
&\approx x^2 - \frac{x^4}{2} + \frac{x^6}{3} - 8x^4 + 4x^6 + \frac{32x^6}{3} + ... \\
&\approx x^2 - \frac{17x^4}{2} + 15x^6
\end{aligned}$$

(e) (4 points) The error of the Taylor series of $f(x)$ centered at $a = 0$ in the interval $x \in [0, 0.1]$ is bounded by 10^{-7}. Using the Taylor approximation in Part (d), find the numerical value of

$$f(0.1) = \ln(1 + 0.01)\cos(0.4)$$

with an accuracy of 6 decimal digits, i.e. at least 6 decimal digits correct.

Because the error is bounded by 10^{-7}, the approximation in Part (d) will be accurate to 6 decimal places. Thus,

$$\begin{aligned} f(0.1) &= \ln(1 + 0.01)\cos(0.4) \\ &\approx (0.1)^2 - \frac{17(0.1)^4}{2} + 15(0.1)^6 \\ &\approx 0.0091650... \end{aligned}$$

The actual value is $f(0.1) = 0.0091649...$

2. (20 points) Consider the three points $P = (2, 1, 0)$, $Q = (3, 0, 4)$ and $R = (-5, 1, -1)$.

(a) (4 points) Explain why the vectors $u = \vec{PQ}$ and $v = \vec{PR}$ are

$$u = \langle 1, -1, 4\rangle, \quad v = \langle -7, 0, -1\rangle.$$

The vectors $u = \vec{PQ}$ and $v = \vec{PR}$ are given by subtracting the first point from the second.

$$\begin{aligned} u &= \langle 3-2, 0-1, 4-0\rangle = \langle 1, -1, 4\rangle \\ v &= \langle -5-2, 1-1, -1-0\rangle = \langle -7, 0, -1\rangle \end{aligned}$$

(b) (4 points) Compute $\cos\theta$, where θ is the angle between u and v.

We use both definitions of the dot product:

$$\begin{aligned} u \cdot v &= 1(-7) + (-1)(0) + 4(-1) \\ &= -7 - 4 \\ &= -11 \end{aligned}$$

We use the geometric definition next: $u \cdot v = |u||v| * \cos(\theta)$.

$$\begin{aligned} |u| &= \sqrt{1^2 + (-1)^2 + 4^2} = \sqrt{18} \\ |v| &= \sqrt{(-7)^2 + (0)^2 + (-1)^2} = \sqrt{50} \\ u \cdot v &= \sqrt{50 * 18}\ \cos(\theta) = 30\ \cos(\theta) \end{aligned}$$

Equating the two definitions of the dot product,

$$\begin{aligned} -11 &= 30\ \cos(\theta) \\ \Rightarrow\ \cos(\theta) &= -\frac{11}{30} \\ \Rightarrow \theta &\approx 111.5^\circ \end{aligned}$$

(c) (4 points) Calculate the cross product $u \times v$ and find the area of the parallelogram spanned by u and v.

We use the determinant version of the cross product.

$$\begin{aligned} u \times v &= \begin{vmatrix} i & j & k \\ 1 & -1 & 4 \\ -7 & 0 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 4 \\ 0 & -1 \end{vmatrix} i + \begin{vmatrix} 1 & 4 \\ -7 & -1 \end{vmatrix} j + \begin{vmatrix} 1 & -1 \\ -7 & 0 \end{vmatrix} k \\ &= \langle 1, -27, -7 \rangle \end{aligned}$$

The area of the parallelogram spanned by u and v is given by the magnitude of the cross product.

$$|\langle 1, -27, -7 \rangle| = \sqrt{1^2 + (-27)^2 + (-7)^2} = \sqrt{779}$$

(d) (4 points) Find an equation for the plane π that contains the points P, Q, R.

We use the normal vector calculated in part (c), $\hat{n} = \langle 1, -27, -7 \rangle$, in the plane equation, ax + by + cz = d, along with the point (2,1,0).

$$\begin{aligned} x - 27y - 7z &= d \\ (2) - 27(1) - 7(0) &= d \\ -25 &= d \end{aligned}$$

Thus, $\pi = \{x - 27y - 7z = -25\}$.

(e) (4 points) What is the distance from the point $S = (1, 0, 1)$ to the plane π?

Using the point on the plane $P = (2, 1, 0)$, we calculate the vector between P and S as $\vec{PS} = \langle 1, 1, -1 \rangle$. The normal vector to the plane π is given by $n = \langle 1, -27, -7 \rangle$ with magnitude $\sqrt{779}$. Then using the equation for the distance to a plane,

$$
\begin{aligned}
d &= \frac{|\vec{PS} \cdot n|}{|n|} \\
&= \left| \langle 1, 1, -1 \rangle \cdot \frac{\langle 1, -27, -7 \rangle}{\sqrt{779}} \right| \\
&= \frac{1}{|}\sqrt{779}(1 - 27 + 7)| \\
&= \frac{19}{\sqrt{779}}
\end{aligned}
$$

3. (20 points) Consider a particle moving according to the trajectory

$$r(t) = \langle e^{-t}\sin(t), e^{-3t}\cos(t), t^5 + t + 3\rangle.$$

(a) (5 points) Find the velocity $v(t)$ of the particle and its acceleration $a(t)$.

We find velocity by taking one time derivative and acceleration by taking another time derivative.

$$\begin{aligned} v(t) &= \frac{d}{dt}\langle e^{-t}\sin(t), e^{-3t}\cos(t), t^5 + t + 3\rangle \\ &= \langle e^{-t}(\cos(t) - \sin(t)), -e^{-3t}(\sin(t) + 3\cos(t), 5t^4 + 1\rangle \\ a(t) &= \frac{d}{dt}v(t) \\ &= \frac{d}{dt}\langle e^{-t}(\cos(t) - \sin(t)), -e^{-3t}(\sin(t) + 3\cos(t), 5t^4 + 1\rangle \\ &= \langle -2e^{-t}\cos(t), e^{-3t}(6\sin(t) + 8\cos(t)), 20t^3\rangle \end{aligned}$$

(b) (5 points) What is the speed of the particle at $t = 0$?

Recalling that speed is a scalar, we find the speed at $t = 0$ by computing the magnitude of $v(0)$.

$$\begin{aligned} v(0) &= \langle 1, -3, 1\rangle \\ |v(0)| &= \sqrt{(1)^2 + (-3)^2 + (1)^2} \\ &= \sqrt{11} \end{aligned}$$

(c) (5 points) Consider the sphere $S = \{x^2 + (y-1)^2 + (z-3)^2 = 25\}$. Decide whether the particle is *inside*, *on* or *outside* of the sphere S at the times $t = 0$ and $t = 10$.

We begin by calculating the particle's location at $t = 0$ and $t = 10$.

$$r(0) = (0, 1, 3)$$
$$r(10) = (-2.5 * 10^{-5}, -7.9 * 10^{-14}, 100013)$$

We then plug these points, (x_0, y_0, z_0) into the equation for the sphere:
$x_0^2 + (y_0 - 1)^2 + (z_0 - 3)^2 < 25 \Rightarrow$ the point is inside the sphere
$x_0^2 + (y_0 - 1)^2 + (z_0 - 3)^2 > 25 \Rightarrow$ the point is outside of the sphere
$x_0^2 + (y_0 - 1)^2 + (z_0 - 3)^2 = 25 \Rightarrow$ the point is on the surface of the sphere.

$t = 0 : (0)^2 + (1 - 1)^2 + (3 - 3)^2 = 0 < 25$
$\Rightarrow r(0)$is inside the sphere (at the center of the sphere!)
$t = 10 : (-2.5 * 10^{-5})^2 + (-7.9 * 10^{-14} - 1)^2 + (100013 - 3)^2 = 1.0 * 10^10 > 25$
$\Rightarrow r(10)$is outside the sphere

(d) (5 points) Compute the distance from the initial position $r(0)$ of the particle to the line given by the intersection of the two planes $\{x = 0\}$ and $\{y = 2\}$.

We first find the direction of the line of intersection of the two planes where $\{x = 0\}$ has normal vector $\langle 1, 0, 0\rangle$ and $\{y = 2\}$ has normal vector $\langle 0, 1, 0\rangle$ by crossing the two vectors.

$$\begin{aligned} v &= \langle 1, 0, 0\rangle \times \langle 0, 1, 0\rangle \\ &= \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \langle 0, 0, 1\rangle \end{aligned}$$

We then find a point S on the line of intersection. We know $x = 0$ and $y = 2$ and we can choose any value for z. One such point is $S = (0, 2, 0)$. Then the vector between $P = r(0) = (0, 1, 3)$ and S is given by $\vec{PS} = \langle 0, 1, -3\rangle$. Using the equation for the distance to a line,

$$\begin{aligned} d &= \left| \vec{PS} \times \frac{v}{|v|} \right| \\ &= |\langle 0, 1, -3\rangle \times \langle 0, 0, 1\rangle| \\ &= |\langle 1, 0, 0\rangle| \\ &= 1 \end{aligned}$$

4. (20 points) Consider the function

$$f(x, y) = x^2y^2 - x^2 - y^2 + 5.$$

(a) (5 points) Compute the partial derivatives $\partial_x f$ and $\partial_y f$.

We compute $\partial_x f$ by taking the derivative of f with respect to x and holding y constant. Similarly, we compute $\partial_y f$ by taking the derivative of f with respect to y and holding x constant.

$$\partial_x f = 2xy^2 - 2x$$
$$\partial_y f = 2x^2y - 2y$$

(b) (5 points) Find all the critical points of $f(x, y)$.

To find critical points we must find all the points where $\partial_x f = 0 = \partial_y f$. We set $\partial_x f = 0$ first:

$$\begin{aligned} 0 &= 2xy^2 - 2x \\ 0 &= 2x(y^2 - 1) \\ &\Rightarrow x = 0, y = \pm 1 \end{aligned}$$

Plugging these values individually into $\partial_y f$ yields

$$\begin{aligned} \partial_y f = 0 &= 2x^2y - 2y \\ x = 0 : -2y &= 0 \\ \Rightarrow y &= 0 \\ y = 1 : 2x^2 - 2 &= 0 \\ \Rightarrow x &= \pm 1 \\ y = -1 : -2x^2 + 2 &= 0 \\ \Rightarrow x &= \pm 1 \end{aligned}$$

Therefore, we have 5 critical points: (0,0), (1,1), (-1,1), (-1,1), (-1,-1).

(c) (5 points) Compute the second partial derivatives $\partial_{xx} f$, $\partial_{yy} f$, $\partial_{xy} f$, and $\partial_{yx} f$.

$$\begin{aligned} \partial_{xx} f &= 2y^2 - 2 \\ \partial_{yy} f &= 2x^2 - 2 \\ \partial_{xy} f &= 4xy \\ \partial_{yx} f &= 4xy \end{aligned}$$

We notice that $\partial_{xy} f$ and $\partial_{yx} f$ are equal as expected.

(d) (5 points) Classify all the critical points of $f(x, y)$ into minima, saddles or maxima.

We calculate the expression $d = f_{xx} f_{yy} - f_{xy}^2$ and analyze its sign after evaluating at each critical point using Theorem 11 from chapter 14.

$$\begin{aligned} & d = f_{xx} f_{yy} - f_{xy}^2 = (2y^2 - 2)(2x^2 - 2) - 16x^2y^2 \\ & d|_{(1,1)} = -16 < 0 \Rightarrow \textit{saddle} \\ & d|_{(-1,1)} = -16 < 0 \Rightarrow \textit{saddle} \\ & d|_{(1,-1)} = -16 < 0 \Rightarrow \textit{saddle} \\ & d|_{(-1,-1)} = -16 < 0 \Rightarrow \textit{saddle} \\ & d|_{(0,0)} = 4 > 0 \rightarrow f_{xx}|_{(0,0)} = -2 < 0 \Rightarrow \textit{maximum} \end{aligned}$$

5. (20 points) For each of the five sentences below, circle the correct answer. There is a unique correct answer per item. (You do *not* need to justify your answer.)

(a) (4 points) The series $\sum_{n=1}^{\infty} \frac{n^2 - 3n + 4}{n^\alpha + n - 5}$ converges

(1) if $\alpha > 3$. (2) if $\alpha \geq 3$. (3) if $\alpha \leq 3$. (4) converges for all α.

(1)

$\sum_{n=1}^{\infty} \frac{n^2-3n+4}{n^\alpha+n-5} \leq \sum_{n=1}^{\infty} \frac{n^2}{n^\alpha}$.

Then by the p-series test, the series will converge if $\alpha - 2 > 1 \Rightarrow \alpha > 3$

(b) (4 points) The value of the infinite series $\sum_{n=0}^{\infty} \frac{5}{7^n}$ is

(1) 7/6 (2) 6/7 (3) 35/6 (4) 5/7 (5) 5/6 (6) ∞

(3)

The series can be rewritten as $\sum_{n=0}^{\infty} \frac{5}{7^n} = \sum_{n=0}^{\infty} 5(\frac{1}{7})^n$ which is a geometric series with $a = 5$ and $r = \frac{1}{7}$. Then $S = \frac{5}{1-\frac{1}{7}} = \frac{35}{6}$.

(c) (4 points) The intersection of $\{x+2y-3z = 10\}$ with $\{(x-4)^2+(y-3)^2+z^2 \leq 10\}$ is

(1) a circle (2) a disk (3) a plane minus a disk (4) two points (5) empty

(2)

Notice that the center of the sphere, (4,3,0), satisfies the equation of the plane,

$$(4) + 2(3) - 3(0) = 10$$

. Thus, the plane passes through the center of the solid sphere so the intersection will be a disk.

(d) (4 points) The midpoint between $P = (-2, 3, -1)$ and $Q = (5, 4, -6)$ is

(1) $(3, 7, -7)$ (2) $(3/2, 7/2, -7/2)$ (3) $(-7, -1, 5)$ (4) $(7, 1, -5)$ (5) $(7/2, 1/2, -5/2)$

(2)

To find the midpoint we compute $\frac{P+Q}{2} = \frac{1}{2}(-2+5, 3+4, -1+(-6)) = \frac{1}{2}(3, 7, -7) = (3/2, 7/2, -7/2)$

(e) (4 points) The radius of convergence of the Taylor expansion of $\ln(1+x)-\ln(1-x)$ is

(1) 0 (2) 1/2 (3) 1 (4) 2/3 (5) $\pi/2$ (6) ∞

(3)

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$
$$\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...$$
$$\ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + ... = \sum_{n=1}^{\infty} 2\frac{x^{2n-1}}{2n-1}$$
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2x^{2(n+1)-1}}{2(n+1)-1}\frac{2n-1}{2x^{2n-1}}\right| < 1$$
$$= \lim_{n\to\infty}\left|x^2 \cdot \frac{2n-1}{2n+1}\right| < 1$$
$$\Rightarrow |x^2| < 1$$
$$\Rightarrow R = 1$$