University of California Davis
Calculus MAT 21C

Name (Print): ______________________
Student ID (Print): ______________________

Practice Midterm II
Time Limit: 50 Minutes

April 28 2023

This examination document contains 8 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Consider the sequence $a_n = \dfrac{8^n}{n!}$ for $n \geq 1$.

   (a) (5 points) Write the first 5 terms of the sequence.

   The first 5 terms are

   $$\frac{8^1}{1!}, \frac{8^2}{2!}, \frac{8^3}{3!}, \frac{8^4}{4!}, \frac{8^5}{5!},$$

   You can leave them like that, or write them as

   $$8, 32, \frac{256}{3}, \frac{512}{3}, \frac{4096}{15}.$$

   (b) (5 points) Justify that the sequence (a_n) is neither increasing nor decreasing.

   By the first part above, the sequence is not decreasing, as $a_1 \leq a_2$ since $8 < 32$. The sequence cannot be increasing either because $a_7 = a_8$.

   Alternatively, a_n is not increasing because $n!$ grows faster than 8^n. Therefore it must *eventually* occur that the sequence a_n starts decreasing. Hence it is not increasing either. You can also just notice that $a_8 > a_{16}$ for instance, by comparing orders of magnitude: $a_8 \approx 400$ and $a_{16} \approx 10$.

(c) (10 points) Argue that (a_n) is a convergent sequence.

It suffices to notice that $n!$ grows much faster than any exponential (such as 8^n) in the hierarchy of functions. Thus a_n converges and in fact $a_n \to 0$.

Alternatively, one can use the Monotone Convergence Theorem by showing that a_n is eventually decreasing and bounded below. It is bounded below because $0 \leq a_n$ for all n. It is eventually decreasing because

$$a_{n+1} < a_n \iff \frac{8^{n+1}}{(n+1)!} < \frac{8^n}{n!} \iff \frac{8^{n+1}}{8^n} < \frac{(n+1)!}{n!} \iff 8 < n.$$

So a_n starts decreasing at $n = 8$. By the Monotone Convergence Theorem, an eventually decreasing sequence bounded below must converge.

(d) (5 points) Show that $\lim_{n\to\infty} a_n = 0$.

As said above, $n!$ grows faster than 8^n in the hierarchy of growth and thus $a_n \to 0$.

Alternatively, we can argue with the Comparison Theorem (the so-called *squeezing* or *sandwich* theorem) by noticing that $0 \leq a_n$ and that a_n is bounded above by

$$\frac{8^n}{n!} = \frac{8^n}{n \cdot (n-1) \cdot \ldots \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} =$$

$$= \frac{8 \cdot 8^{n-8} \cdot 8^7}{n \cdot (n-1) \cdot \ldots \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \leq \frac{8^8}{n \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \to 0$$

as it was similarly argued in lecture.

2. (25 points) Let us consider the series

$$S = \sum_{n=1}^{\infty} (-1)^n \cdot \left(\sqrt[n]{3} - 1 \right).$$

(a) (5 points) Show that the sequence $b_n = \sqrt[n]{3} - 1$, $n \geq 1$, converges to 0.

This is equivalent to showing that $\sqrt[n]{3} \to 1$ when $n \to \infty$. Indeed,

$$\lim_{n\to\infty} \sqrt[n]{3} = 3^{\lim_{n\to\infty} \frac{1}{n}} = 0.$$

or one may take the logarithm and show it goes to 0:

$$\lim_{n\to\infty} \ln(\sqrt[n]{3}) = \lim_{n\to\infty} \frac{1}{n} \ln(3) = \ln(3) \cdot \lim_{n\to\infty} \frac{1}{n} = \ln(3) \cdot 0 = 0.$$

(b) (10 points) Argue that $b_{n+1} \leq b_n$ for $n \in \mathbb{N}$ large enough.

The inequality $b_{n+1} \leq b_n$ is equivalent to the inequalities

$$\sqrt[n+1]{3} - 1 \leq \sqrt[n]{3} - 1 \iff \sqrt[n+1]{3} \leq \sqrt[n]{3} \iff 3 \leq 3^{\frac{n+1}{n}} = 3 \cdot 3^{1/n} \iff 1 < 3^{1/n}$$

which is true for all $n \geq 1$ since $1 < 3$.

(c) (5 points) Explain why the series S is convergent.

The alternating series test applies because the series is of the form $\sum_{n=1}^{\infty}(-1)^n \cdot b_n$ with b_n converging to 0 and decreasing.

(d) (5 points) Does the following series converge?

$$\sum_{n=1}^{\infty}(-1)^n \cdot \sqrt[n]{3}.$$

It does not because its sequence of terms $(-1)^n \cdot \sqrt[n]{3}$ does not converge to zero. We saw in lecture that $\sum_{n=1}^{\infty} a_n$ converging implied $a_n \to 0$.

3. (25 points) The goal of this problem is to find the value $\sin(0.2)$ with an error less than 10^{-6}, i.e. so that the first 6 decimal digit are accurate.

   (a) (5 points) Explain why the Taylor expansion of $\sin(x)$ at $a = 0$ is

$$\sin(x) \overset{(x\approx 0)}{=} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

The nth derivative of $sin(x)$ is $(-1)^k \cos(x)$ if n is odd $n = 2k+1$, and $(-1)^k \sin(x)$ if n is even $n = 2k$. In particular $sin^{(4)}(x) = x$ and thus the derivatives have period 4. The derivatives evaluated at 0, starting at $n = 0$ yields the sequence of values $0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1, 0, 1, 0, -1 \ldots$ and the four digits $0, 1, 0, -1$ repeat forever. This yields the Taylor expansion above, where only odd terms appear and the sign is alternating.

   (b) (10 points) Show that the Taylor truncation of degree 5 at $a = 0$ has error bounded by 10^{-6} at $x = 0.2$.

By the Theorem on the Error of the Taylor approximation, the error $R_5(x)$ of the degree 5 approximation is given by

$$R_5(x) = \left| \frac{\sin^{(6)}(c)}{6!}(x-0)^6 \right| = \left| \frac{-\sin(c)}{720} x^6 \right| = \left| \frac{\sin(c)}{720} x^6 \right|,$$

where c is a value between 0 and x. Since $\sin(x)$ is overall bounded by 1, i.e. $|\sin(x)| \leq 1$ for all x, we have the upper bound

$$R_5(x) = \left| \frac{\sin(c)}{720} x^6 \right| \leq \left| \frac{x^6}{720} \right|$$

At $x = 0.2 = 2 \cdot 10^{-1}$ this upper bound is

$$\left| \frac{2^6 \cdot 10^{-6}}{720} \right| \leq 10^{-6}.$$

(c) (5 points) Evaluate $\sin(0.2)$ with the first 5 decimal digits being exactly accurate.

By the previous part, it suffices to take the Taylor truncation at order 5. This is

$$\sin(0.2) \approx 0.2 - \frac{(0.2)^3}{3!} + \frac{(0.2)^5}{5!}.$$

The previous part guarantees that the error is less than 10^{-6} and thus the first 5 digits are correct.

(d) (5 points) Please find the mistake in the following (wrong) argument: *the error in the Taylor series trucated at degree* 5 *is given by the next term in the Taylor series. Therefore, the error is given by*

$$\frac{f^{(6)}(0)}{6!}x^6 = \frac{sin^{(6)}(0)}{6!}x^6 = \frac{-sin(0)}{6!} = 0.$$

In conclusion, the Taylor truncation of degree 5 *actually approximates* $\sin(x)$ *with no error. Thus* $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ *exactly.*

The mistake is in the sentence "the error in the Taylor series trucated at degree 5 is given by the next term in the Taylor series.". This is not true, the error is given by a term involving the next derivative evaluated at c, where c an undetermined point, but it is **not** the next term in the Taylor series.

4. (25 points) For each of the ten sentences below, circle the correct answer. (You do *not* need to justify your answer.)

(a) (5 points) The value of the infinite series $\sum_{n=0}^{\infty}(-1)^n \cdot \left(\frac{3}{2^n}\right)$ is

(1) 0. (2) 1 **(3)** 2 (4) 3 (5) ∞.

(b) (5 points) The Taylor expansion of $\ln(1+x^2)$ at $x=0$ of order 6 is

(1) $x - \frac{x^2}{2} + \frac{x^3}{3}$. (2) $x + \frac{x^2}{2} + \frac{x^3}{3}$ **(3)** $x^2 - \frac{x^4}{2} + \frac{x^6}{3}$ (4) $x^2 - \frac{x^4}{4} + \frac{x^6}{6}$.

(c) (5 points) The ratio test applied the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$

(1) concludes convergence. (2) concludes divergence (3) is inconclusive

(d) (5 points) The value of the geometric series $\sum_{n=0}^{\infty} 5^n$ is

(1) 0 (2) -0.25 (3) 0.25 (4) 0.5 **(5)** ∞

(e) (5 points) If a sequence (a_n) converges then (a_n) is bounded below.

(1) True. (2) False.