This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. Fill in all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may not use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this and explain why the theorem may be applied.

(B) Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) Mysterious or unsupported answers will not receive full credit. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

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1. (25 points) Consider the sequence \((a_n)\) defined recursively by \(a_{n+1} = \sqrt[3]{a_n^2 + a_n + 2}\) and initial condition \(a_1 = 0\).

(a) (15 points) Show that the sequence \((a_n)\) is convergent.

We will argue that \((a_n)\) is increasing and bounded above. By the Monotone Convergence Theorem, this will prove that \((a_n)\) converges.

**Increasing.** For the first two terms we have \(a_1 = 0\), \(a_2 = \sqrt[3]{2}\) and thus \(a_1 < a_2\). In general, we want to argue that \(a_n < a_{n+1}\). Note that all the terms \(a_n\) are positive and greater than 1 (because \(\sqrt[3]{2}\) is greater than 1). The cubic power is increasing, therefore \(a_n < a_{n+1}\) if and only if \(a_n^3 < a_{n+1}^3\), which is the inequality

\[
a_n^3 < a_n^2 + a_n + 2.
\]

Since the function \(x^3 - x^2 - x - 2\) is increasing for \(x \geq 1\), the inequality is true for all \(a_n\) and thus \(a_n < a_{n+1}\).

**Bounded** Let us argue that \(a_n \leq 2\). This is true for \(a_1 = 2\). In general, if we have iteratively argued that \(a_n \leq 2\) we need to show that \(a_{n+1} \leq 2\) as well. Indeed

\[
a_{n+1} = \sqrt[3]{a_n^2 + a_n + 2} \leq \sqrt[3]{2^2 + 2 + 2} = \sqrt[3]{8} = 2.
\]

(b) (10 points) Find the limit of the sequence \((a_n)\).

Let \(L = \lim_{n \to \infty} a_n\), then \(a_{n+1} = \sqrt[3]{a_n^2 + a_n + 2}\) implies that \(L\) must satisfy \(L = \sqrt[3]{L^2 + L + 2}\). The only real solution of this equation is \(L = 2\), therefore \(L = 2\).
2. (25 points) Solve the two parts below

(a) (20 points) For each of the series below, determine whether the series converges or diverges. You must justify your answer in detail. If you are applying a certain test, state the name of the test clearly, the steps implementing the test and its outcome. If a sequence converges, you do not need to find the limit.

1. \[ \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{12^n} , \]
2. \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5 + 3n + 6}} , \]
3. \[ \sum_{n=1}^{\infty} \frac{\ln(n)}{n^n} , \]
4. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} . \]

For (1), the series is absolutely convergent (and so convergent) because

\[ \left| \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{12^n} \right| \leq \sum_{n=1}^{\infty} \frac{1}{12^n} \]

and \[ \sum_{n=1}^{\infty} \frac{1}{12^n} \] converges because it is geometric with \( r = 1/12 < 1 \). The direct comparison test implies that \[ \sum_{n=1}^{\infty} \frac{\cos(\ln(n))}{12^n} \] (absolutely) converges.

For (2), the direct comparison test shows that \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5 + 3n + 6}} \] converges if and only if \[ \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \]. The latter does converge because it is a \( p \)-harmonic series with \( p = 5/2 \) and this is less than 1 (or e.g. via the integral test).

For (3), the root test states that this series converges if the limit \( \sqrt[n]{\left| \frac{\ln(n)}{n^n} \right|} \) exists and it is less than 1. Since

\[ \lim_{n \to \infty} \sqrt[n]{\left| \frac{\ln(n)}{n^n} \right|} = \lim_{n \to \infty} \frac{\sqrt[n]{\ln(n)}}{n} = \lim_{n \to \infty} \frac{1}{n} = 0 , \]

since \( \lim_{n \to \infty} \sqrt[n]{\ln(n)} = 1 \). Since \( 0 < 1 \), the root test proves that the series converges.

For (4), the alternating series test does apply because \( \frac{(-1)^n}{\sqrt{n}} \) is decreasing, positive and converging to 0. Therefore the series converges.
(b) (5 points) Discuss for which positive real values of \( \alpha \in (0, \infty) \) this series converges:

\[
\sum_{n=1}^{\infty} \frac{n^3}{1 + n^\alpha}.
\]

The direct comparison test shows that this series converges if and only if

\[
\sum_{n=1}^{\infty} \frac{n^3}{n^\alpha} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha-3}}
\]

converges. This latter series is a \( p \)-harmonic series with \( p = \alpha/3 \). Thus, the series converges if and only if \( \alpha - 3 > 1 \). In consequence, the original series converges if and only if \( \alpha > 4 \). In conclusion, for \( \alpha \in (0, 4] \) the series diverges and for \( \alpha \in (4, \infty) \) it converges.
3. (25 points) Solve the following parts.

(a) (8 points) Find the Taylor expansion of $x^2 \cos(x)$ of order 8 at $x = 0$.

The Taylor expansion of $\cos(x)$ at $x = 0$ is

$$\cos(x) \equiv_{x=0} 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$ 

Therefore, the Taylor expansion of $x^2 \cos(x)$ at $x = 0$ is

$$x^2 \cos(x) \equiv_{x=0} x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \frac{x^8}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!}.$$ 

The expansion at order 8 is thus

$$x^2 - \frac{x^4}{2!} + \frac{x^6}{4!} - \frac{x^8}{6!}.$$ 

(b) (7 points) Find the Taylor expansion of $f(x) = x^2 \cos(x^3)$ of order 20 at $x = 0$.

By Part (a), the Taylor expansion of $\cos(x^3)$ at $x = 0$ is

$$\cos(x^3) \equiv_{x=0} 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}.$$ 

Therefore, the Taylor expansion of $x^2 \cos(x^3)$ at $x = 0$ is

$$x^2 \cos(x^3) \equiv_{x=0} x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(2n)!}.$$ 

Hence, the Taylor expansion of $f(x) = x^2 \cos(x^3)$ of order 20 at $x = 0$ is

$$x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!}.$$
(c) (7 points) Compute the radius of convergence of the Taylor series of \( f(x) \) at \( x = 0 \).

The series \( \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{2n!} \) converges for all \( x \in (-\infty, \infty) \). For instance, this is implied by the ratio test applied to this series. In consequence, the radius of convergence is \( R = \infty \), i.e. the series converges in the entire real line.

(d) (3 points) What is the approximated value of \( f(0.1) \) given by the Taylor approximation of order 20 at \( x = 0.1 \)?

By Part (c) it suffices to evaluate

\[
x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!}
\]

at \( x = 0.1 \). Therefore the answer is

\[
(0.1)^2 - \frac{(0.1)^8}{2} + \frac{(0.1)^{14}}{24} - \frac{(0.1)^{20}}{720}.
\]

(For your information, this value is approximately 0.00999999500000041... It is fine to present the answer as the expression above.)
4. (25 points) For each of the ten sentences below, circle whether they are true or false. (You do not need to justify your answer.)

(a) (5 points) If a sequence \((a_n)\) converges to 0, then \(\sum_{n=1}^{\infty} a_n\) converges.

(1) True.  (2) False.

(b) (5 points) If a power series \(A(x)\) (centered at \(x = 0\)) diverges for \(x = 12\), then it diverges for \(x = -14\).

(1) True.  (2) False.

(c) (5 points) The Taylor series of a non-constant differentiable function \(f: \mathbb{R} \rightarrow \mathbb{R}\) at \(x = 0\) cannot have all its Taylor coefficients be equal to zero.

(1) True.  (2) False.

(d) (5 points) The Taylor series associated to a real polynomial centered at any point \(x = a, \ a \in (-\infty, \infty)\), always converges.

(1) True.  (2) False.

(e) (5 points) If the ratio test can determine the convergence of a convergent series \(\sum_{n=1}^{\infty} a_n\), then so can the root test.

(1) True.  (2) False.