University of California Davis **Name (Print):** ____________________
Algebraic Topology MAT 215B **Student ID (Print):** ____________________

Midterm Examination **May 3 2024**
Time Limit: Due 5/6@3pm

This examination document contains 6 pages, including this cover page, and 3 problems.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	30	
2	40	
3	30	
Total:	100	

1. (30 points) Compute the homology $H_*(X)$ of the following two topological spaces X.
   (a) (15 points) Let $p \in S^n \times S^n$. Then $X = (S^n \times S^n) \setminus \{p\}$.

   (b) (15 points) Let $C_1, C_2 \subseteq \mathbb{R}^3$ be the two circles

   $$C_1 = \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2 = 1, z = 0\}, \quad C_2 = \{(x, y, z) \in \mathbb{R}^3 : (y-1)^2+z^2 = 1, x = 0\}.$$

   Then $X = \mathbb{R}^3 \setminus (C_1 \cup C_2)$.

2. (40 points) Let $A_\bullet, B_\bullet$ be two chain complexes and $f : A_\bullet \to B_\bullet$ a chain map. Consider the chain complex $C(f)_\bullet := A_{\bullet-1} \oplus B_\bullet$ with differential

$$\partial_C(\alpha, \beta) := (-\partial_A \alpha, \partial_B \beta - f(\alpha) = \begin{pmatrix} -\partial_A & 0 \\ -f & \partial_B \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

(a) (10 points) Show that $\partial_C^2 = 0$, and so $(C(f), \partial_C)$ is indeed a chain complex.

(b) (10 points) Prove that f is a quasi-isomorphism iff $C(f)$ is exact.
(Here exact means $(C(f), \partial_C)$ has vanishing homology, a.k.a. $C(f)$ is acyclic.)

(c) (10 points) Let $f, g : A_\bullet \to B_\bullet$ be homotopic $f \simeq g$. Show that $C(f)$ and $C(g)$ are quasi-isomorphic, i.e. $\exists$ a chain map $F : C(f) \to C(g)$ inducing an isomorphism in homology.

(d) (10 points) Suppose that f is a homotopy equivalence. Show that the identity map $\mathrm{id} : C(f) \to C(f)$ is null-homotopic (i.e. homotopic to the zero chain map).

3. (30 points) Solve the following parts.

   (a) (10 points) Give an example of two spaces X, Y such that their homologies are isomorphic $H_*(X) \cong H_*(Y)$ but $X \not\simeq Y$ are not homotopy equivalent.

   (b) (10 points) Give an example of a surjective map $f : X \to Y$ such that the induced map $H_*(f) : H_*(X) \to H_*(Y)$ is not surjective.

(c) (10 points) Show that $X = S^2 \times S^2$ and $Y = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ have the same homology groups and the same homotopy groups.

(d) (Extra 10 points) Show that $X = S^2 \times S^2$ and $Y = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ are not homotopy equivalent.