MAT 215B: PROBLEM SET 2

DUE TO FRIDAY MAY 3 2024

Abstract. This problem set corresponds to the first week of the course MAT-215B Spring 2024. It is due Friday May 3 at 9:00pm submitted via Gradescope.

Task: Solve *two* of the problems below and submit it through Gradescope by Friday May 3 at 9pm. Be rigorous and precise in writing your solutions.

Exercises in topological aspects

Problem 1. Compute the relative homology of the following pairs (X, A):

(1) $X = S^1$ and $A = \{p_1, p_2, \ldots, p_n\}$, where $p_i \in S^1$ are (pairwise) distinct points.

(2) $X = S^1 \times [0, 1]$ and $A = \partial X$.

(3) Problem 17.(a) of Section 2.1 in the textbook.

Problem 2. Find two topological spaces X, Y such that these two conditions simultaneously hold:

(1) $X \not\simeq Y$, i.e. X, Y are not homotopic.

(2) $H_*(X) \cong H_*(Y)$, i.e. the singular homologies of X, Y coincide as $\mathbb{Z}$-modules.

Problem 3. Problems 20 and 21 in the textbook (in Section 2.1).

Problem 4. Compute the singular homology $H_*(X)$ for each of the following topological spaces. In choosing the spaces X for this exercise, I am trying to it make a point: explore each of this spaces and see if you can find it.

(a) $X = \{(z_1, z_2, z_3, z_4) : z_1 z_2 z_3 z_4 = 0\} \subseteq \mathbb{C}^4$.

(b) $X = \{(z_1, z_2, z_3, z_4, z_5) : z_1^2 + z_2^3 + z_3^7 + z_4 z_5^3 = 0\} \subseteq \mathbb{C}^5$.

(c) $X = \{(z_1, z_2, z_3, z_4, z_5) : |z_1|^2 + z_2^3 + |z_3|^9 + |z_4| z_5^3 = 0\} \subseteq \mathbb{C}^5$.

(d) $X = \{(x, y, z, w) \in \mathbb{R}^4 : 5x^3y^3 + xy^9 - 2y^{12} + x^2z^6 + 7y^8w^2 = 0\} \subseteq \mathbb{R}^4$.

(e) $X = \{A \in M_n(\mathbb{R}) : a_{ij} = 0 \text{ if } i < j\}$ where $M_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$ denotes the space of $(n \times n)$-matrices $A = (a_{ij})$, $1 \leq i, j \leq n$, with real entries $a_{ij} \in \mathbb{R}$.

(f) Let V be a finite-dimensional $\mathbb{R}$-vector space. Let X be the space of all inner products $\langle \cdot, \cdot \rangle : V \otimes V \longrightarrow \mathbb{R}$.

(g) Let $V = \mathbb{R}^\infty$ be the infinite-dimensional $\mathbb{R}$-vector space. Let $X = \mathbb{R}^\infty \setminus \{0\}$ be the set of its non-zero vectors.

The next two bonus items will not be graded. They might be of interest if you are also taking the Analysis Sequence, or as a matter of general interest.

(h) (Bonus I) Let H be a Hilbert space. For instance, let us choose

$$H = \{(z_1, z_2, \ldots) : \sum_{i=1}^{\infty} |z_n|^2 \text{ convergent}\} \subseteq \mathbb{C}^{\infty},$$

the set of all infinite sequences of complex numbers, or $H = \mathbb{C}^{\infty}$. Choose $X = H \setminus \{0\}$.

(i) (Bonus II) Let $X = \{f \in L^2(\mathbb{C}, \mathbb{R}) : \left(\int_{\mathbb{C}} |f|^2\right)^{1/2} = 1\}$, where the space $L^2(\mathbb{C}, \mathbb{R})$ of L^2-integrable functions $f : \mathbb{C} \longrightarrow \mathbb{R}$ is given the topology induced by the inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}dz.$$

Problem 5. Solve the following problems.

(a) Let $X_2 = \{l \subseteq \mathbb{R}^3 : l \text{ is an oriented line}\}$ be the space of oriented lines in $\mathbb{R}^3$. Show that $H_0(X_2) = H_2(X_2) \cong \mathbb{Z}$ and $H_i(X_2) = 0$ if $i \in \mathbb{N}$, $i \neq 0, 2$.

(b) Let $X_3 = \{l \subseteq \mathbb{R}^4 : l \text{ is an oriented line}\}$ be the space of oriented lines in $\mathbb{R}^4$. Show that $H_0(X_3) = H_3(X_3) \cong \mathbb{Z}$ and $H_i(X_3) = 0$ if $i \in \mathbb{N}$, $i \neq 0, 3$.

(c) Let $\mathrm{SL}(2, \mathbb{C}) = \{A \in M_2(\mathbb{C}) : \det(A) = 1\} \subseteq M_2(\mathbb{C})$ be the space of (2×2)-matrices with complex entries and determinant one. Show that we have $H_0(\mathrm{SL}(2, \mathbb{C})) \cong \mathbb{Z}$, $H_3(\mathrm{SL}(2, \mathbb{C})) \cong \mathbb{Z}$ and $H_i(\mathrm{SL}(2, \mathbb{C})) = 0$ if $i \in \mathbb{N}$, $i \neq 0, 3$.

Exercises in homological algebra

Many parts of the following problems were discussed in detail in class (especially for the first three). The purpose of solving them is that you review the material and perform all the computations needed by yourself.

Problem 1. Let (A_*, ∂^A) and (B_*, ∂^B) be two chain complexes of Abelian groups, indexed by $* \in \mathbb{N}$. Let $i_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ be a map of chain complexes, i.e. $i_{*-1} \circ \partial^A = \partial^B \circ i_*$. Show that $i_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ descends to a map

$$H_*(i) : (A_*, \partial^A) \longrightarrow (B_*, \partial^B),$$

between the homologies of these chain complexes.

Problem 2. Let $f, g : X \longrightarrow Y$ be two continuous maps between topological spaces X, Y, which we assume to be CW complexes. Let $(C_\bullet(X), \partial^X)$ be the singular chain complex for X and $(C_\bullet(Y), \partial^Y)$ be the singular chain complex for Y.

(a) Show that $f_\# : (C_\bullet(X), \partial^X) \longrightarrow (C_\bullet(Y), \partial^Y)$ satifies

$$f_\# \circ \partial^X = \partial^Y \circ f_\#.$$

In particular, $f_\#$ descends to a map in homology.

(b) Suppose that $f \simeq g$ and $F : X \times [0, 1] \longrightarrow Y$ is a homotopy between f and g. Show that there exists a *chain homotopy*, constructed using F, between the two maps $f_\#$ and $g_\#$ of chain complexes.

(c) Show that $X \simeq Y$ implies that their singular homology groups $H_*(X) \cong H_*(Y)$ are all isomorphic.

Problem 3. Prove the Five-Lemma (as stated in Section 2.1 of textbook).

Problem 4. Let (A_*, ∂^A) and (B_*, ∂^B) be two chain complexes of Abelian groups, indexed by $* \in \mathbb{N}$. A map of chain complexes $f_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ is said to be a quasi-isomorphism if $H_*(f)$ are isomorphisms.

A map of chain complexes $f_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ is said to be a chain homotopy equivalence if there exists $g_* : (B_*, \partial^B) \longrightarrow (A_*, \partial^A)$ such that $f_* \circ g_* \simeq id$ and $g_* \circ f_* \simeq id$ are chain homotopic to the identity.

(a) Show that a chain homotopy equivalence is a quasi-isomorphism.

(b) Give an example of a chain map $f_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ which is a quasi-isomorphism but *not* a chain homotopy equivalence.

(c) Given an example of a chain complex (A_*, ∂^A) whose homology groups are zero and such that $id_A : (A_*, \partial^A) \longrightarrow (A_*, \partial^A)$ is not homotopic to the zero map.

Problem 5. Let $(A_*, \partial^A), (B_*, \partial^B)$ and (C_*, ∂^C) be three chain complexes of Abelian groups, indexed by $* \in \mathbb{N}$. Let $i_* : (A_*, \partial^A) \longrightarrow (B_*, \partial^B)$ and $j_* : (B_*, \partial^B) \longrightarrow (C_*, \partial^C)$ be two maps of chain complexes, i.e. $i_{*-1} \circ \partial^A = \partial^B \circ i_*$ and $j_{*-1} \circ \partial^B = \partial^C \circ j_*$.

Suppose that i is an injective map, i.e. i_* is injective for all $* \in \mathbb{N}$, j is a surjective map, and $im(i_*) = \ker(j_*)$.

(a) Show that the image of $H_*(i) : H_*(A_*, \partial^A) \longrightarrow H_*(B_*, \partial^B)$ equals the kernel of $H_*(j) : H_*(B_*, \partial^B) \longrightarrow H_*(C_*, \partial^C)$.

(b) For each $n \in \mathbb{N}$, let $c_n \in C_n$ be an n-cycle, and consider *any* element $b_n \in B_n$ such that $j(b_n) = c_n$. Show that there exists an element $a_{n-1} \in A_{n-1}$ such that $i_{n-1}(a_{n-1}) = \partial^B b_n$.

(c) Let us now define the map

$$\delta_* : H_*(C_*, \partial^C) \longrightarrow H_{*-1}(A_*, \partial^A), \quad \delta([c_n]) = [a_{n-1}],$$

where a_{n-1} is *any* element obtained as in Part (b). Here $[c] \in H_n(X)$ denotes the homology class of a cycle $c \in C_n(X)$. Show that $\delta_n([c_n]) = [a_{n-1}]$ is independent of the choice of b_n and a_{n-1} in Part (b), and thus it is a well-defined map.

(d) Show that $im(j_*) = \ker(\delta_*)$ as subgroups of $H_*(C_*, \partial^C)$.

(e) Show that $im(\delta_*) = \ker(i_{*-1})$ as subgroups of $H_{*-1}(A_*, \partial^A)$.