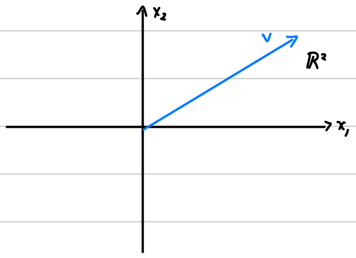
Linear maps (6.1 & 6.6)

Recall: V, W vector spaces. By def,

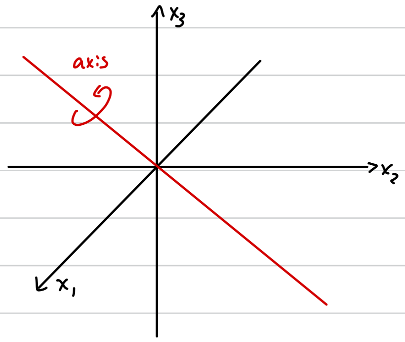
$f: V \to W$ linear map $\iff \begin{cases} ① \ f(v_1+v_2) = f(v_1) + f(v_2) \quad \forall v_1, v_2 \in V \\ ② \ f(a \cdot v) = a \cdot f(v) \quad \forall v \in V, a \in \mathbb{R} \end{cases}$

Ex. (i) $f(v) = v$ $\forall v$ is linear: call it the identity

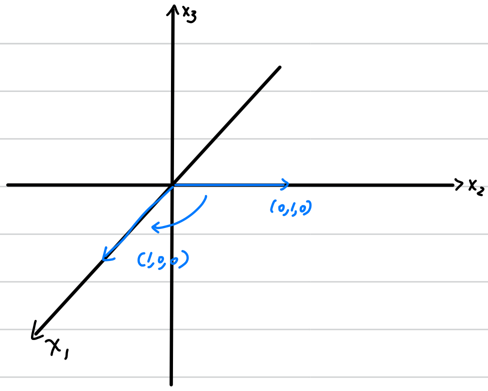
$f: V \to V$



(ii) Rotations along axis (through the origin) are linear



(iii) Permutations of axis



(iv) NON-LINEAR:

translation is not LINEAR

How to understand linear maps? $f: V \to W$

Q1. If $w \in W$ given, describe all $v \in V$ s.t. $f(v) = w$ ← goal

Q.2. If $f: V \to W$, $g: W \to Z$, understand $g \circ f : V \to Z$

$f \circ g \neq g \circ f$ if $V = W = Z$

NO!!!

Applications of Q1 & Q2:

① Differential eq^n: e.g. $F = ma \iff F(x(t)) = m \cdot x''(t)$ ← Q.1.

② Quantum mechanics: Heisenberg's uncertainty principle ← Q.2.

↳ position & momentum non commutative

f linear map $\quad g$ linear map

IMPORTANT: from $f: V \xrightarrow{\text{Linear}} W$ to "matrix"

⚠ we need a choice: basis of V & basis of W

basis are ordered collections

Same linear map $\xrightarrow[\text{choices}]{\text{different}}$ different matrices

Theorem: (6.1.3)

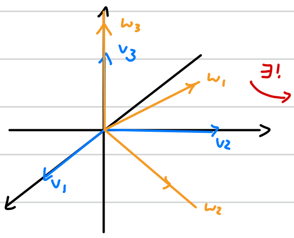
Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be basis of V & W. Then $\exists!$ $f: V \to W$ linear s.t. $f(v_i) = w_i$, $\forall i \in \{1, \dots, n\}$

unique

exists

Example: $V = W = \mathbb{R}^3$, $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$

$w_1 = (1,1,0)$, $w_2 = (3,2,0)$, $w_3 = (0,0,7)$



$\exists!$ $f(x_1, x_2, x_3) = (x_1 + x_2, 3x_1 + 2x_2, 7x_3)$

but different basis will make writing formulas complicated

Proof of theorem:

need to prove existance & uniqueness

we are given $v \in V$: what is $f(v)$?

$\exists a_i \in \mathbb{R}$ s.t. $v = a_i v_1 + \dots + a_n v_n$ & unique

$$\therefore f(v) = f(a_1 v_1 + \dots + a_n v_n) \overset{\text{Linearity}}{=} f(a_1 v_1) + \dots + f(a_n v_n)$$

$$= a_1 f(v_1) + \dots + a_n f(v_n) \quad \text{(linearity)}$$

$$= a_1 w_1 + \dots + a_n w_n \quad \blacksquare$$

by what we want of f