Linear maps to matrices

V, W v.s. ① $f: V \xrightarrow{\text{linear}} W$
② $\{v_i\}$ basis of V
③ $\{w_j\}$ basis of W

unique matrix A_f → $\dim(W)$ rows, $\dim(V)$ columns ($W \times V$ matrix)

Example:

$f: \mathbb{R}^3 \xrightarrow{\text{linear}} \mathbb{R}^2$

$\{v_i\} = \{(1,0,0), (0,1,1), (1,1,2)\}$
$\{w_j\} = \{(3,0), (2,1)\}$

$\rightarrow A_f = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix}$ 2×3 matrix

Def: Linear maps to matrix

Let V, W be v.s., $f: V^n \to W^m$ & $\{v_i\}$, $\{w_j\}$ basis of V, W. The matrix A_f associated to f in the basis $\{v_i\}, \{w_j\}$

$$A_f = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \quad m \times n \text{ matrix}$$

where $f(v_i) = a_{1i} w_1 + \dots + a_{mi} w_m$ ($\exists!\, a_{ji} \in \mathbb{R}$)

Example

$f: \mathbb{R}^3 \xrightarrow{\text{linear}} \mathbb{R}^2$

$\{v_i\} = \{(1,0,0), (0,1,1), (1,1,2)\}$

$\{w_j\} = \{(3,0), (2,1)\}$

choose $f(x_1, x_2, x_3) = (x_1 - 3x_2, x_2 + x_3)$

$v_1 \longmapsto f(v_1) = (1,0) = a_{11}(3,0) + a_{21}(2,1)$
$= \frac{1}{3}(3,0) + 0(2,1)$

$v_2 \longmapsto f(v_2) = (-3,2) = a_{12}(3,0) + a_{22}(2,1)$
$= -\frac{7}{3}(3,0) + 2(2,1)$

$v_3 \longmapsto f(v_3) = (-2,3) = a_{13}(3,0) + a_{32}(2,1)$
$= -\frac{8}{3}(3,0) + 3(2,1)$

$$A_f = \begin{pmatrix} \frac{1}{3} & -\frac{7}{3} & -\frac{8}{3} \\ 0 & 2 & 3 \end{pmatrix}$$

multiplication with $v \in V$ yields coordinates in Basis coordinates

Ex. 2. Different bases for same map give different matrices!

$V = W = \mathbb{R}^3$, $\{v_i\} = \{w_j\}$ for this example

First choice: $\{v_i\} = \{(1,0,0), (0,1,0), (0,0,1)\}$
$\{w_j\} = \{(1,0,0), (0,1,0), (0,0,1)\}$

A_f = 3x3 matrix ($f: \mathbb{R}^3 \to \mathbb{R}^3$)

$f(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, 6x_1 - x_2, -x_1 - 2x_2 - x_3)$

$$A_f = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

Second choice : $\{v_i\} = \{(-1,-6,13),(-2,-3,2),(-1,2,1)\} = \{w_j\}$

$$\left.\begin{aligned} v_1 &\longmapsto f(v_1) = (0,0,0) \quad a_{i1} = 0 \\ v_2 &\longmapsto f(v_2) = (-6,-9,6) = 0 \cdot w_1 + 3w_2 + 0w_3 \\ v_3 &\longmapsto f(v_3) = (4,-8,-4) = 0 \cdot w_1 + 0 \cdot w_2 - 4w_3 \end{aligned}\right\} \quad A_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{pmatrix}$$