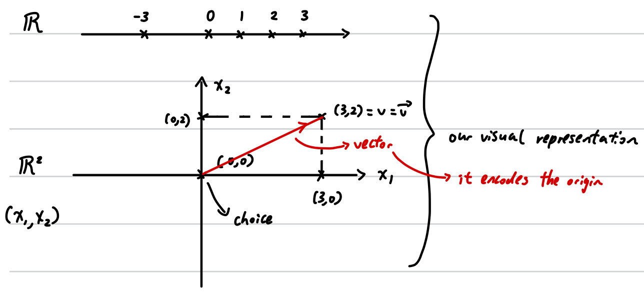
The geometry of linear maps (drawing & visualizing)

Linear maps: $f: \mathbb{R}^n \to \mathbb{R}^m$ such that ① $f(x+y) = f(x) + f(y)$;

② $f(c \cdot x) = c \cdot f(x) \quad \forall c \in \mathbb{R}$



Example: Understand all linear maps $f: \mathbb{R} \to \mathbb{R}$

$x \mapsto f(x)$

Verify that f is linear $\Rightarrow$ $f(x)$ must be of the form $\boxed{f(x) = \alpha x}$ for some $\alpha \in \mathbb{R}$

In fact, $f: \mathbb{R} \to \mathbb{R}$ f linear $\Leftrightarrow$ f is of the form $\boxed{f(x) = \alpha x}$ (some $\mathbb{R}$ # (fixed))

(hint: $\alpha = f(1)$)

Proof: Let $\alpha = f(1)$

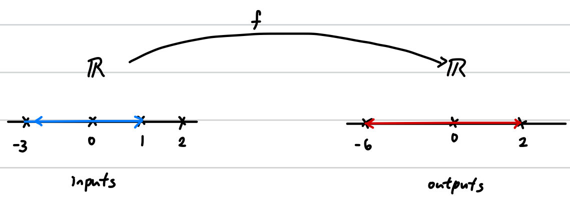
$\because$ $f(x)$ is linear

$f(k) = k f(1) = k\alpha$

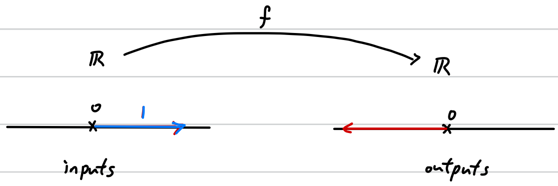
$\therefore$ $f(x)$ must take the form of $f(x) = \alpha x$

ℝ:

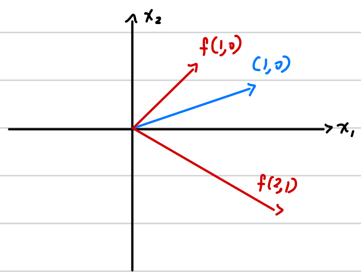
(1) $f(x) = 2x$, $(\alpha = 2)$



(2) $f(x) = -7x$ $(\alpha = -7)$



ℝ²



$f: \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (3x_1 + x_2, x_1 - 4x_2)$

$(0,0) \longmapsto f(0,0) = (0,0)$

$(1,0) \longmapsto f(1,0) = (3,1)$

$(2,1) \longmapsto f(2,1) = (4,-2)$

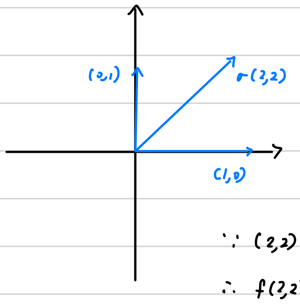
Lemma (useful)

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is linear, then any image $f(x_1, x_2)$ is uniquely determined by the images $\boxed{f(1,0), f(0,1)}$ → bases

Proof: $\because$ f is linear,

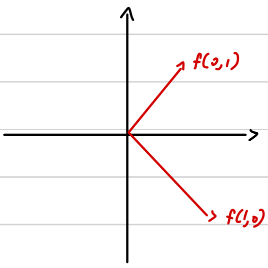
$$f(x_1, x_2) = f(\alpha(1,0), \beta(0,1)) = \alpha f(1,0) + \beta f(0,1) \qquad (\alpha, \beta \in \mathbb{R})$$

$\therefore$ $f(x_1, x_2)$ is uniquely determined by the images $f(1,0)$, $f(0,1)$



$\because$ $(2,2) = 2(1,0) + 2(0,1)$

$\therefore$ $f(2,2) = 2f(1,0) + 2f(0,1)$



Example: ("Rotations in $\mathbb{R}^2$") Fix an angle $\theta \in \mathbb{R}$. Consider the map $\mathbb{R}_\theta: \mathbb{R}^2 \to \mathbb{R}^2$

$$R_\theta(x_1, x_2) = (\cos\theta x_1 - \sin\theta x_2, \ \sin\theta x_1 + \cos\theta x_2)$$

$$\mathbb{R}_{\frac{\pi}{2}}(x_1, x_2) = (-x_2, x_1)$$