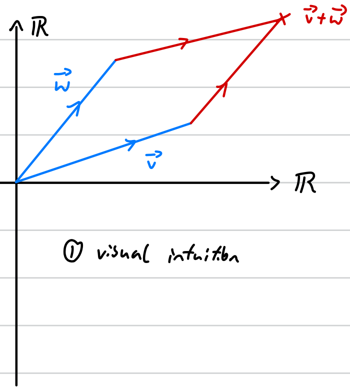
Vector spaces

What are the 2 operations

that allow us to define linearity?



What does $\vec v+\vec w$ satisfy (vector addition)

① $\vec v+\vec w = \vec w+\vec v$ (commutativity)

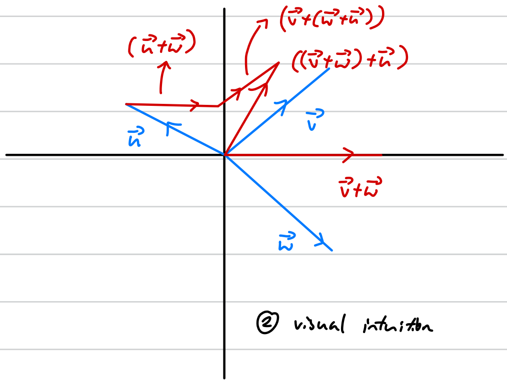
② $(\vec v+\vec w)+\vec u = \vec v+(\vec w+\vec u)$

③ $\vec v+\vec 0 = \vec v \leftrightsquigarrow \exists \vec 0$ such that $\boxed{\text{neutral element}}$

④ $\vec v+(-\vec v) = \vec 0 \leftarrow \exists(-\vec v)$ $\boxed{\text{additive inverse}}$

↑ exists ($\exists!$ ← exists and unique)

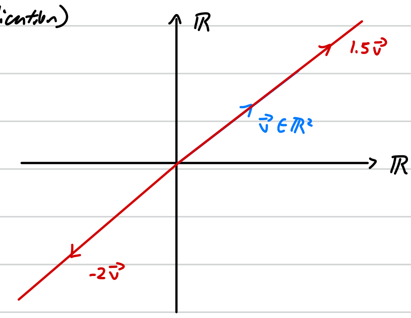
Exercise: Prove ③ & ④ $\exists!$



What does scalar must satisfy (scalar multiplication)

① $c\cdot(\vec v+\vec w) = c\vec v + c\vec w$ (distributivity)

② $\exists\, 1$ such that $1\cdot\vec v = \vec v$



Def: (vector space)

A vector space V over field K is a set V endowed with 2 operations:

① $V\times V \to V$ (ie. $(\vec v,\vec w)\longmapsto \vec v+\vec w$)

② $K\times V \to V$ (ie. $(c,\vec v)\longmapsto c\cdot\vec v$)

such that properties of vector addition & scalar multiplication are satisfied (see book def. 4.11.)

Examples:

I.

① $V=\mathbb{R}^n$, $\vec{v}=(x_1,\dots,x_n)\in\mathbb{R}^n$, $\vec{w}=(y_1,\dots,y_n)\in\mathbb{R}^n$ $\Big\}$ $\vec{v}+\vec{w}:=(x_1+y_1,\dots,x_n+y_n)$

② $K=\mathbb{R}$, $c\in\mathbb{R}$, $\vec{v}=(x_1,\dots,x_n)\in\mathbb{R}^n$ $\Big\}$ $c\cdot\vec{v}:=(cx_1,\dots,cx_n)$

$\boxed{\text{This is a vector space}}$

II.

① $V=\mathbb{Q}^n$ rationals

② $K=\mathbb{Q}$

$\Big\}$ addition & multiplication are satisfied gives that $\boxed{\mathbb{Q}^n \text{ is a } \mathbb{Q}\text{-vector space}}$

III.

① $V=\mathbb{Q}$ $\quad \vec{q_1}+\vec{q_2}$ is fine but $c\cdot\vec{q}\notin V$

$K=\mathbb{R}$ $\quad$ $\mathbb{R}$ $\quad$ $\mathbb{Q}$ $\quad$ $\therefore$ $\boxed{\mathbb{Q} \text{ is not a vector space}}$

IV.

① $V=\{\text{all polynomials } p(x)=a_0+a_1x+\dots+a_nx^n\}$ for $a_i\in\mathbb{R}$, for some $n\in\mathbb{N}$

Vectors $\vec{v}\in V$ are polynomials

$v=1-x+3x^2$, $w=x^2-x^4$ $\Big\}$ $v+w=1-x+4x^2-x^4$

② $c\in K=\mathbb{R}$

$c\cdot(1-x+3x^2)=c\cdot v=c-cx+3cx^2$

$\boxed{\text{The polynomials are vector spaces}}$

V.

$V=\{\text{all polynomials of degree at most } n\}$

$\boxed{\text{is a } \mathbb{R} \text{ vector space.}}$

$V=\{\text{all polynomials of degree exactly } n\}$

$\boxed{\text{is NOT a } \mathbb{R}\text{-vector space}}$

Contradiction,

$v=1-x^2$, $w=3+x+x^2$ $\Big\}$ $v+w=4+x$

No neutral element as 0 is not allowed