Sums & Intersections of subspaces

Recap: ① Linear systems of eq^n

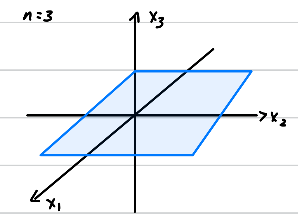
② Linear maps $f: \mathbb{R}^n \to \mathbb{R}^m$

③ geometry of linear maps

④ Vector spaces: V an $\mathbb{R}$-v.s.

⑤ vector subspaces: $\underbrace{U_1, U_2}_{\mathbb{R}\text{-v.s. subspace of } V} \leq V$

Ex. $V = \mathbb{R}^n = \left\{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\}$. A vector $\vec{v} \in \mathbb{R}^n$ is $\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$



$U \leq \mathbb{R}^3 = V$ given by

$U = \{U \in V : x_3 = 0\} \leq V$ is a subspace (a plane)

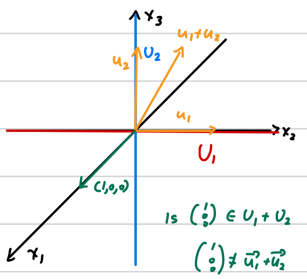
Def^n (Sum of subspaces)

Let V be an $\mathbb{R}$-v.s. & $U_1, U_2 \leq V$ subspace. By def^n, the sum subspace $U_1 + U_2$ is given by:

$U_1 + U_2 = \{\vec{u_1} + \vec{u_2} : \vec{u_1} \in U_1, \vec{u_2} \in U_2\} \leq V$

i.e. $\{v \in V : \vec{v} = \vec{u_1} + \vec{u_2}$ for some $\vec{u_1} \in U_1$ & $\vec{u_2} \in U_2\}$

Ex. $V = \mathbb{R}^3$, $U_1 = \{x_1 = x_3 = 0\}$, $U_2 = \{x_1 = 0, x_2 = 0\}$



What is $U_1 + U_2 \leq V = \mathbb{R}^3$

$\hookrightarrow U_1 + U_2 = \{x_1 = 0\}$

$= (x_2, x_3)$-plane $= \{(0, x_2, x_3) \in \mathbb{R}^3\}$

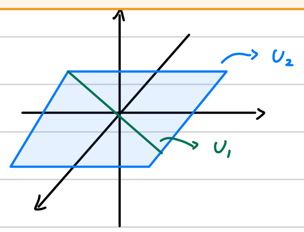
is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in U_1 + U_2$ No

$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \vec{u_1} + \vec{u_2}$

Remark:

(i) $U_1 + U_2$ is a v.s., furthermore $U_1 \leq U_1 + \overset{\text{can be } \vec{0}}{U_2}$ & $U_2 \leq U_1 + \overset{\text{can be } \vec{0}}{U_2}$

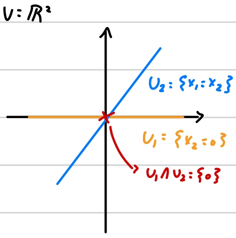
(ii) Typically $U_1 + U_2 \overset{?}{\neq} U_1, U_2$, But if $U_1 \leq U_2$, then $U_1 + U_2 = U_2$



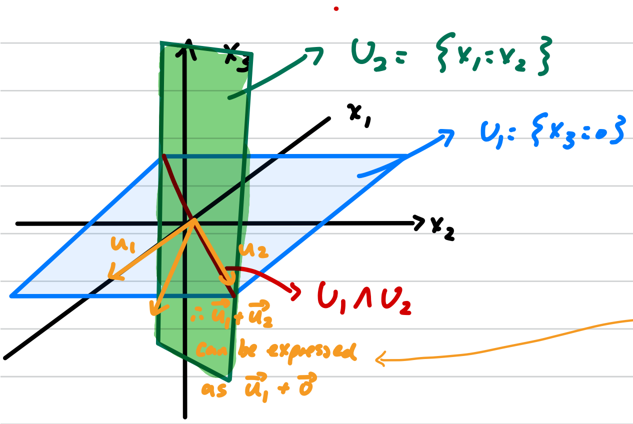
Def (intersection of subspaces)

V an $\mathbb{R}$-v.s., U_1, U_2 subspaces. Then $U_1 \cap U_2 := \{ v \in V : u \in U_1 \text{ and } v \in U_2 \}$

Ex. $V = \mathbb{R}^2$



$V = \mathbb{R}^3$



Motivation for direct sums:

$V = \mathbb{R}^3$, $U_1 = \{x_3 = 0\}$

$U_2 = \{x_1 = x_2\}$

Then $U_1 + U_2 = V = \mathbb{R}^3$

any vector $v \in \mathbb{R}^3$ is of the form $\vec{v} = \vec{u}_1 + \vec{u}_2$

BUT NOT UNIQUE

Def^n: (direct sum)

V an $\mathbb{R}$-v.s., $U_1, U_2 \subseteq V$. Then V is set to be a $\boxed{\text{direct sum}}$ of U_1 & U_2 if:

(1) $V = U_1 + U_2$, (V is a sum of U_1 & U_2)

(2) Any vector $v \in V$ can be *uniquely* expressed as $\vec{v} = \vec{u}_1 + \vec{u}_2$, $\vec{u}_i \in U_i$

we write
$V = U_1 \oplus U_2$
$\oplus$ indicates direct sum

Proposition (4.47)

V a v.s., $U_1, U_2 \subseteq V$ subspaces. Then

$V = U_1 \oplus U_2$ is a direct sum $\iff$ $V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$

Proof: ($\Rightarrow$)

$\underbrace{\text{We assume } V = U_1 \oplus U_2}$. We want $\overset{V = U_1 + U_2 \checkmark}{U_1 \cap U_2 = \{0\}}$

$V = U_1 + U_2 \checkmark$

$\vec{v} = \vec{u}_1 + \vec{u}_2$ then u_1, u_2 are unique

need to check

By contradiction:

Suppose $\vec{w} \in U_1 \cap U_2$, $\vec{w} \neq 0$. Now choose any $\vec{v} \in V$ and the unique decomposition $\vec{v} = \vec{u}_1 + \vec{u}_2$

but then $\vec{v} = \underbrace{(\vec{u}_1 + \vec{w})}_{\neq u_1} + \underbrace{(\vec{u}_2 - \vec{w})}_{\neq u_2}$ ($\because \vec{w} \in U_1 \cap U_2$

$\therefore (\vec{u}_1 + \vec{w}) \in U_1$

gives a different decomposition $(\vec{u}_2 - \vec{w}) \in U_2$)

Proof: ($\Leftarrow$)

We assume $U_1 \cap U_2 = \{0\}$ and $V = U_1 + U_2$, we want $\underbrace{V = U_1 \oplus U_2}$

need to check $\quad V = U_1 + U_2$ ✓

$\vec{v} = \vec{u_1} + \vec{u_2}$ then u_1, u_2 are unique

Assume there exists $U_1 \cap U_2 = \{0\}$ where the decomposition $\vec{v} = \vec{u_1} + \vec{u_2}$ is not unique (for any $\vec{v} \in V$), ie.

$\vec{v} = \vec{u_1} + \vec{u_2}$ and $\vec{v} = \vec{w_1} + \vec{w_2}$ where $\vec{u_1}, \vec{w_1} \in U_1$ & $\vec{u_2}, \vec{w_2} \in U_2$, $u_1 \neq w_1$ & $u_2 \neq w_2$

Thus, $\vec{u_1} + \vec{u_2} = \vec{w_1} + \vec{w_2}$

$\vec{u_1} - \vec{w_1} = \vec{w_2} - \vec{u_2}$ is a non-zero in $U_1 \cap U_2$ //