University of California Davis **Name (Print):** ____________________
Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Practice Midterm Examination **April 26 2024**
Time Limit: 50 Minutes

This examination document contains 7 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

Do not write in the table to the right.

1. (25 points) Let $V = \mathbb{R}^4$ and consider the two subsets

$$U_1 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + 2x_3 + x_4 = 0\},$$

$$U_2 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 3x_1 - x_2 + 8x_3 = 0, x_4 = 0\}.$$

(a) (5 points) Show that $U_1 \subseteq V$ is a vector subspace.

Solution. Let $f : V \to \mathbb{R}$ be given by $f(x_1, x_2, x_3, x_4) = x_1 + x_2 + 2x_3 + x_4 = 0$. Then $U_1 = \{v \in V : f(v) = 0\}$. Since f is a linear function,

$$f(v_1 + v_2) = f(v_1) + f(v_2) = 0, \quad \forall v_1, v_2 \in U_1,$$

$$f(a \cdot v_1) = a \cdot f(v_1) = 0, \quad \forall v_1 \in U_1.$$

Therefore $U_1 \subseteq V$ is a subspace, as it is closed under sum and scalar multiplication.

(b) (10 points) Show that $V = U_1 + U_2$.

Solution. The vectors $v_1 = (1, -1, 0, 0)$, $v_2 = (0, 1, 0, -1)$ and $v_3 = (-2, 0, 1, 0)$ are linearly independent. Since $U_1 \neq V$, $\{v_1, v_2, v_3\}$ are a basis for U_1. It suffices to show that $U_1 \neq U_2$ to conclude $V = U_1 + U_2$. Indeed, $w = (1, 3, 0, 0) \in U_2$ but $w \notin U_1$, since it does not satisfy the defining equation for U_1. Thus $\{v_1, v_2, v_3, w\}$ are linearly independent, and so they are a basis for V. In particular, $U_1 + U_2 = V$.

(c) (5 points) Find a vector $v \in V$ such that $U_1 \cap U_2 = \mathrm{span}(v)$.

Solution. It suffices to solve the equations for U_1 and U_2 simultaneously:

$$x_1 + x_2 + 2x_3 + x_4 = 0$$
$$3x_1 - x_2 + 8x_3 = 0$$
$$x_4 = 0$$

For instance, the vector $v = (10, -2, -4, 0)$ belongs to both U_1 and U_2 and it is non-zero. So we conclude $U_1 \cap U_2 = \mathrm{span}(v)$.

(d) (5 points) Prove that $V \neq U_1 \oplus U_2$.

Solution. $V = U_1 \oplus U_2$ if and only if $V = U_1 + U_2$ and $V = U_1 \cap U_2 = \{0\}$. By Part (c), $V = U_1 \cap U_2 = \mathrm{span}(v) \neq \{0\}$, so V is no a direct sum of U_1 and U_2.

2. (25 points) Consider the vector space $V = \mathbb{R}^3$ and the vectors

$$v_1 = (3, 4, -1), \quad v_2 = (0, 2, 5), \quad v_3 = (-6, -2, 17).$$

(a) (10 points) Decide whether v_3 is a linear combination of $\{v_1, v_2\}$.

Solution. We can try to write $v_3 = a_1 v_1 + a_2 v_2$. Since the first component of v_2 is zero, this forces $a_1 = -2$. Then $a_2 = 3$ gives $-2v_1 + 3v_2 = v_3$.

(b) (5 points) Is v_1 a linear combination of $\{v_2, v_3\}$?

Solution. Yes, since v_3 is a linear combination of v_1, v_2, then $v_1 \in \text{span}(v_2, v_1) = \text{span}(v_2, v_3)$.

(c) (5 points) Show that $(1, 0, 0)$ is *not* a linear combination of $\{v_1, v_2, v_3\}$.

Solution. Since $v_3 \in \text{span}(v_1, v_2)$, $(1, 0, 0)$ being a linear combination of $\{v_1, v_2\}$ would imply that there exists $a_1, a_2 \in \mathbb{R}$ such that

$$(1, 0, 0) = a_1 v_1 + a_2 v_2.$$

There is no solutions for these equations so $(1, 0, 0) \notin \text{span}(v_1, v_2)$.

(d) (5 points) Find a vector $w \in V$ such that $V = \text{span}(v_1, v_2) \oplus \text{span}(w)$.

Solution. We can directly take $w = (1, 0, 0)$.

3. (25 points) Consider $V = \mathbb{R}^n$ for some $n \in \mathbb{N}$. Solve the following parts.

   (a) (20 points) Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear function which is not equal to the zero function. Show that the subset

$$U_f := \{v \in V : f(v) = 0\}$$

   is a vector subspace.

   Solution. As in Problem 1, since f is a linear function,

$$f(v_1 + v_2) = f(v_1) + f(v_2) = 0, \quad \forall v_1, v_2 \in U_f,$$

$$f(a \cdot v_1) = a \cdot f(v_1) = 0, \quad \forall v_1 \in U_f.$$

   Therefore $U_1 \subseteq V$ is a subspace, as it is closed under sum and scalar multiplication.

   (b) (5 points) Show that there exists a $w \in V$ such that $V = U \oplus \text{span}(w)$.

   Solution. Note that $U_f \neq V$, as there is a non-zero vector $w \in V$ such that $w \notin U_1$ because f is not the zero function. Since $\dim(U_f) = n - 1$, we must have $V = U \oplus \text{span}(w)$ for this particular w.

4. (25 points) For each of the sentences below, circle whether they are **true** or **false**. (You do *not* need to justify your answer.)

   (a) (5 points) Let $V = \mathbb{R}[x]$. Then $U = \{p(x) \in V : p(0) = 1\}$ is a vector subspace.

   (1) True. (2) **False**.

   (b) (5 points) Let $V = \mathbb{R}[x]$. Then $U = \{p(x) \in V : p(0) = p(1)\}$ is a vector subspace.

   (1) **True**. (2) False.

   (c) (5 points) If $v \in V$ is a vector, there exist different vectors $w_1, w_2 \in V$ such that $v + w_1 = 0$ and $v + w_2 = 0$.

   (1) True. (2) **False**.

   (d) (5 points) The function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

   $$f(x_1, x_2, x_3) = (x_1 - 2x_2 + 6x_3, x_1 + 4x_2 - 1)$$

   is a linear function.

   (1) True. (2) **False**.

   (e) (5 points) The function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by

   $$f(x_1, x_2) = (x_1, x_2, 3x_1x_2, x_1 + x_2^4)$$

   is a linear function.

   (1) True. (2) **False**.