University of California Davis **Name (Print):** ____________________
Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Practice Midterm Examination 2 **April 26 2024**
Time Limit: 50 Minutes

This examination document contains 9 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^3$ and consider the vectors

$$v_1 = (3, 2, 0), \quad v_2 = (1, 1, 1), \quad v_3 = (6, -5, 1), \quad v_4 = (1, 0, 0).$$

Define the subspaces $U_1 := \operatorname{span}(v_1, v_2, v_3)$, $U_2 = \operatorname{span}(v_1, v_2)$ and $U_3 = \operatorname{span}(v_3, v_4)$.

(a) (10 points) Show that $V = U_1$.

(b) (5 points) Show that $V = U_2 + U_3$.

(c) (5 points) Prove or disprove whether $V = U_2 \oplus U_3$.

(d) (5 points) Find two vectors $w_1, w_2 \in V$ such that $V = \mathrm{span}(v_4, w_1, w_2)$.

2. (25 points) Consider the vector space $V = \mathbb{R}[x]$ and the vectors

$$p_1(x) = 1 - x^2 + 3x^5, \quad p_2(x) = x + x^3, \quad p_3(x) = 1 - 4x - x^2 - 4x^3 + 3x^5.$$

(a) (10 points) Show that the subset $U = \{p(x) \in \mathbb{R}[x] : p(2) = 0\}$ is a vector subspace.

(b) (5 points) Prove that $p_3(x) \in \operatorname{span}(p_1(x), p_2(x))$.

(c) (5 points) Show that the intersection

$$\mathrm{span}(p_1(x), p_2(x), p_3(x)) \cap U \neq= \{0\}$$

contains at least a non-zero polynomial.

(d) (5 points) For each n, find a subspace $W_n \subseteq U$ such that $\dim(W_n) = n$.

3. (25 points) Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$f(x_1, x_2, x_3) = (x_1 + x_2, 3x_1 - x_2 + 2x_3).$$

(a) (10 points) Show that the subset

$$U_f := \{v \in V : f(v) = 0\}$$

is a vector subspace.

(b) (5 points) Is the subset

$$\{v \in V : f(v) = 1\}$$

a vector subspace? (Justify your answer.)

(c) (5 points) Consider the vector $w = (1, -1, -2) \in \mathbb{R}^3$. Show that $w \in U_f$.

(d) (5 points) Show that $U_f = \operatorname{span}(w)$.

4. (25 points) Consider the vector space $V = \mathbb{R}^5$ and the subspaces

$$U_1 := \{(x_1, x_2, x_3, x_4, x_5) \in V : x_1 - x_2 + 3x_4 - 6x_5 = 0\},$$

$$U_2 := \operatorname{span}(v_1, v_2, v_3),$$

where $v_1 = (1, 0, -1, 0, 1)$, $v_2 = (4, 1, 0, 1, 1)$ and $v_3 = (0, 0, 1, 1, 0)$.

(a) (10 points) Show that $\{v_2, v_1 + \frac{5}{3}v_3\}$ is a basis for the subspace $U_1 \cap U_2 \subseteq V$.

(b) (5 points) Find a basis for the subspace $U_1 \subseteq V$.

(c) (5 points) Show that $V = U_1 \oplus \text{span}(v_1)$.

(d) (5 points) Prove that $V \neq U_1 \oplus \text{span}(v_2)$. Is it true that $V = U_1 \oplus \text{span}(v_3)$?