University of California Davis **Name (Print):** ____________________

Abstract Linear Algebra MAT 67 **Student ID (Print):** ____________________

Practice Midterm Examination 2 **April 26 2024**

Time Limit: 50 Minutes

This examination document contains 9 pages, including this cover page, and 4 problems. You must verify whether there any pages missing, in which case you should let the instructor know. **Fill in** all the requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

(A) **If you use a lemma, proposition or theorem which we have seen in the class or in the book, you must indicate this** and explain why the theorem may be applied.

(B) **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive little credit.

(C) **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive little credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

(D) If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Let $V = \mathbb{R}^3$ and consider the vectors

$$v_1 = (3, 2, 0), \quad v_2 = (1, 1, 1), \quad v_3 = (6, -5, 1), \quad v_4 = (1, 0, 0).$$

Define the subspaces $U_1 := \operatorname{span}(v_1, v_2, v_3)$, $U_2 = \operatorname{span}(v_1, v_2)$ and $U_3 = \operatorname{span}(v_3, v_4)$.

(a) (10 points) Show that $V = U_1$.

Solution. Since $U_1 \subseteq V$, it suffices to show $V \subseteq U_1$. Equivalently, that $\{v_1, v_2, v_3\}$ span V (and so they are a basis). It is clear that v_1 and v_2 are linearly independent, as they are not a multiple of each other. Let us show that $v_3 \notin \operatorname{span}(v_1, v_2)$.

By contradiction, if $v_3 \in \operatorname{span}(v_1, v_2)$ then $\exists a_1, a_2 \in \mathbb{R}$ such that $v_3 = a_1 v_1 + a_2 v_2$. Since the third component of v_1 is zero, this forces $a_2 = 1$. But then we must have $v_3 = a_1 v_1 + a_2 v_2$, which is

$$(3a_1, 2a_1, 0) + (1, 1, 1) = (6, -5, 1).$$

There is no a_1 solving this equality, since we would have $3a_1 + 1 = 6$ and $2a_1 + 1 = -5$, a contraction. Therefore $v_3 \notin \operatorname{span}(v_1, v_2)$.

(b) (5 points) Show that $V = U_2 + U_3$.

Solution. By Part (a), $V = U_1$. Since $U_1 \subseteq U_2 + U_3$, we must have $V \subseteq U_2 + U_3$. Conversely, since $U_2, U_3 \subseteq V$, their sum is also a subspace $U_2 + U_3 \subseteq V$. This concludes $V = U_2 + U_3$.

(c) (5 points) Prove or disprove whether $V = U_2 \oplus U_3$.

Solution. It is **not** true that $V = U_2 \oplus U_3$. Since $U_1 = U_2 \oplus \text{span}(v_3)$ equals V, v_4 must be a linear combination of v_1, v_2, v_3. Given that v_4 is not linearly dependent with v_3, it must be that $U_2 \cap U_3 \neq \{0\}$. (This intersection is in fact a line, 1-dimensional.) So V it is not a direct sum of U_2 and U_3.

(d) (5 points) Find two vectors $w_1, w_2 \in V$ such that $V = \text{span}(v_4, w_1, w_2)$.

Solution. There are (infinitely) many choices. For instance, we can take $w_1 = (0, 1, 0)$ and $w_2 = (0, 0, 1)$, the coordinate basis.

2. (25 points) Consider the vector space $V = \mathbb{R}[x]$ and the vectors

$$p_1(x) = 1 - x^2 + 3x^5, \quad p_2(x) = x + x^3, \quad p_3(x) = 1 - 4x - x^2 - 4x^3 + 3x^5.$$

(a) (10 points) Show that the subset $U = \{p(x) \in \mathbb{R}[x] : p(2) = 0\}$ is a vector subspace.

Solution. For any polynomial $p(x) = a_0 + a_1 x + \ldots + a_n x^n$, the equation $p(2) = 0$ is

$$a_0 + 2a_1 + \ldots + 2^n a_n = 0,$$

which is a linear equation on the variables $a_0, \ldots, a_n$. Therefore U is the solution set of a linear homogeneous equation, so it is a vector subspace.

Alternatively, one can check closed under sums and scalar multiplication. For instance, for closed under sums, take $p, q \in U$ so that $p(2) = 0$ and $q(2) = 0$. We want to show that $p + q \in U$. This is true because $(p + q)(2) = p(2) + p(q) = 0$.

(b) (5 points) Prove that $p_3(x) \in \text{span}(p_1(x), p_2(x))$.

Solution. We have the equality $p_3 = p_1 - 4p_2$, so $p_3(x) \in \text{span}(p_1(x), p_2(x))$.

(c) (5 points) Show that the intersection

$$\text{span}(p_1(x), p_2(x), p_3(x)) \cap U \neq \{0\}$$

contains at least a non-zero polynomial.

Solution. We need a polynomial in U, i.e. that has a root equal to 2, and that it is a linear combination of p_1, p_2, p_3. Since Part (b) implies that $p_3(x) \in \text{span}(p_1(x), p_2(x))$, it suffices to look for linear combinations of p_1 and p_2. We want $a_1, a_2 \in \mathbb{R}$ such that $a_1p_1 + a_2p_2$ has 2 as a root. This is the equation

$$a_1p_1(2) + a_2p_2(2) = 0.$$

We can expand this to

$$a_1(1 - 2^2 + 3 \cdot 2^5) + a_2(2 + 2^3) = 0, \text{ i.e.}$$

$$93a_1 + 10a_2 = 0.$$

Choose any a_1, a_2 with $a_2 = -9.3a_1$, e.g. $a_1 = 10$ and $a_2 = -93$. Then we have $10p_1 - 93p_2 \in U$ and, by construction, also in $\text{span}(p_1, p_2)$.

(d) (5 points) For each n, find a subspace $W_n \subseteq U$ such that $\dim(W_n) = n$.

Solution. Let $v_j = (x - 2)^j$ for $j \in \mathbb{N}$ and choose $W_n = \text{span}(v_1, \ldots, v_n)$.

3. (25 points) Consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ given by

$$f(x_1, x_2, x_3) = (x_1 + x_2, 3x_1 - x_2 + 2x_3).$$

(a) (10 points) Show that the subset

$$U_f := \{v \in V : f(v) = 0\}$$

is a vector subspace.

Solution. Since f is a linear function,

$$f(v_1 + v_2) = f(v_1) + f(v_2) = 0, \quad \forall v_1, v_2 \in U_f,$$

$$f(a \cdot v_1) = a \cdot f(v_1) = 0, \quad \forall v_1 \in U_f.$$

Therefore $U_f \subseteq V$ is a subspace, as it is closed under sum and scalar multiplication.

(b) (5 points) Is the subset

$$\{v \in V : f(v) = 1\}$$

a vector subspace? (Justify your answer.)

Solution. No. For instance, it does not contain a *zero* vector. It is also not closed under sums, nor closed under scalar multiplication.

(c) (5 points) Consider the vector $w = (1, -1, -2) \in \mathbb{R}^3$. Show that $w \in U_f$.

Solution. We need to evaluate $f(w)$, where $f(x_1, x_2, x_3) = (x_1 + x_2, 3x_1 - x_2 + 2x_3)$ and $w = (1, -1, -2)$. We have

$$f(w) = (1 + (-1), 3 \cdot 1 - (-1) + 2 \cdot (-2)) = (0, 0),$$

and so $w \in U_f$.

(d) (5 points) Show that $U_f = \text{span}(w)$.

Solution. By Part (c), $\text{span}(w) \subseteq U_f$ because $w \in U_f$. It suffices to show $U_f \subseteq \text{span}(w)$. Suppose that $v \in U_f$ is given by $v = (x_1, x_2, x_3)$. Then $f(v) = 0$ are the equations

$$x_1 + x_2 = 0, \quad 3x_1 - x_2 + 2x_3 = 0.$$

The first equation implies $x_2 = -x_1$ and the second $4x_1 + 2x_3 = 0$, so that $x_3 = -2x_1$. This implies that $v = a_1 \cdot w$ where $a_1 = x_1$, and thus $v \in \text{span}(w)$. This proves $U_f \subseteq \text{span}(w)$ and thus we conclude $U_f = \text{span}(w)$.

4. (25 points) Consider the vector space $V = \mathbb{R}^5$ and the subspaces

$$U_1 := \{(x_1, x_2, x_3, x_4, x_5) \in V : x_1 - x_2 + 3x_4 - 6x_5 = 0\},$$

$$U_2 := \text{span}(v_1, v_2, v_3),$$

where $v_1 = (1, 0, -1, 0, 1)$, $v_2 = (4, 1, 0, 1, 1)$ and $v_3 = (0, 0, 1, 1, 0)$.

(a) (10 points) Show that $\{v_2, v_1 + \frac{5}{3}v_3\}$ is a basis for the subspace $U_1 \cap U_2 \subseteq V$.

Solution. We need to argue that $\{v_2, v_1 + \frac{5}{3}v_3\}$ are linearly independent first. This is clear, as v_2 is *not* a multiple of $v_1 + \frac{5}{3}v_3$. Now we need to show $\text{span}(v_2, v_1 + \frac{5}{3}v_3) = U_1 \cap U_2$.

For the inclusion $\text{span}(v_2, v_1 + \frac{5}{3}v_3) \subseteq U_1 \cap U_2$, we just check directly that $v_2 \in U_1$ and $v_1 + \frac{5}{3}v_3 \in U_1$. For instance, $v_2 \in U_1$ because $4 - 1 + 3 \cdot 1 - 6 \cdot 1 = 0$.

For the inclusion $U_1 \cap U_2 \subseteq \text{span}(v_2, v_1 + \frac{5}{3}v_3)$. Note that $v_1, v_3 \notin U_1$ and $v_2 \in U_1$. Since U_1 is 4-dimensional and $\text{span}(v_1, v_3)$ is 2-dimensional, $v_1, v_3 \notin U_1$ implies that the intersection $U_1 \cap \text{span}(v_1, v_3)$ is 1-dimensional. Therefore $U_1 \cap U_2$ is 2-dimensional, with a possible basis given by v_2 and any non-zero vector of $U_1 \cap \text{span}(v_1, v_3)$. Since $v_1 + \frac{5}{3}v_3$ is in the intersection, it must be that $\{v_2, v_1 + \frac{5}{3}v_3\}$ is a basis.

(b) (5 points) Find a basis for the subspace $U_1 \subseteq V$.

Solution. By Part (a), we already have 2 linearly independent vectors in U_1. Since U_1 is 4-dimensional, it suffices to give 2 additional vectors $w_1, w_2 \subseteq U_1$ so that $\{v_2, v_1 + \frac{5}{3}v_3, w_1, w_2\}$ are a basis of U_1. Take for instance

$$w_1 = (1, 1, 0, 0, 0), \quad w_2 = (0, 3, 0, 1, 0),$$

both of which are in U_1. A computation shows that $w_1 \notin \text{span}(v_2, v_1 + \frac{5}{3}v_3)$ and $w_2 \notin \text{span}(v_2, v_1 + \frac{5}{3}v_3, w_1)$. Therefore $\{v_2, v_1 + \frac{5}{3}v_3, w_1, w_2\}$ are a basis.

(c) (5 points) Show that $V = U_1 \oplus \mathrm{span}(v_1)$.

Solution. Since U_1 is 4-dimensional[1], any vector $v \in V$ not in U_1 satisfies $V = U_1 + \mathrm{span}(v)$. Since $v \notin U_1$, this is in fact always a direct sum $V = U_1 \oplus \mathrm{span}(v)$. Therefore, it suffices to argue that $v_1 \notin U_1$. This is indeed the case, as

$$1 - 0 + 3 \cdot 0 - 6 \cdot 1 \neq 0,$$

so $v_1 \notin U_1$.

(d) (5 points) Prove that $V \neq U_1 \oplus \mathrm{span}(v_2)$. Is it true that $V = U_1 \oplus \mathrm{span}(v_3)$?

Solution. Since $v_2 \in U_1$, $U_1 \cap \mathrm{span}(v_2) = U_1 \neq \{0\}$ and the sum cannot be a direct sum. Since $v_3 \notin U_1$, the same argument as in Part (c) shows that $V = U_1 \oplus \mathrm{span}(v_3)$.

[1] It is cut out by one non-zero equation in 5-variables.