MAT 67: PROBLEM SET 2

DUE TO FRIDAY APR 19 2024

Abstract. This problem set corresponds to the first week of the course MAT-67 Spring 2024. It is due Friday Apr 19 at 9:00am submitted via Gradescope.

Purpose: The goal of this assignment is to acquire the necessary skills to work with vector spaces. These were discussed during the second week of the course and are covered in Chapter 4 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. Decide whether each of the following sets are $\mathbb{R}$-vector spaces and prove (or disprove) accordingly. Each item is worth 5 points:

(1) The set $\mathbb{R}^n$ with sum and scalar multiplications:
$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n).$$
$$c \cdot (x_1, \ldots, x_n) := (c \cdot x_1, \ldots, c \cdot x_n), \quad \forall c \in \mathbb{R}.$$

(2) Fix a natural number $n \in \mathbb{N}$. The set
$$P_{\leq n} := \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \quad (a_0, \ldots, a_n) \in \mathbb{R}^n\}$$
of polynomials in one variable x of degree at most n with sum
$$(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n) :=$$
$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n$$
and scalar multiplication
$$c \cdot (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) := ca_0 + ca_1 x + ca_2 x^2 + \ldots + ca_n x^n, \quad \forall c \in \mathbb{R}.$$

(3) Fix a natural number $n \in \mathbb{N}$. The set
$$P_n := \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \quad (a_0, \ldots, a_n) \in \mathbb{R}^n, \quad a_n \neq 0\}$$
of polynomials in one variable x of degree exactly n with sum
$$(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n) :=$$
$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n$$
and scalar multiplication
$$c \cdot (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) := ca_0 + ca_1 x + ca_2 x^2 + \ldots + ca_n x^n, \quad \forall c \in \mathbb{R}.$$

(4) The set $\mathbb{Q} \subseteq \mathbb{R}$ of rational numbers with sum and scalar multiplications:
$$q_1 + q_2 := q_1 + q_2, \text{ the usual sum of rational numbers}$$
$$c \cdot q := c \cdot q, \text{ the usual product of a rational number } q \text{ by a real number } c$$

(5) The set $C(\mathbb{R}, \mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } f \text{ is a map}\}$ of maps from $\mathbb{R}$ to $\mathbb{R}$, with sum given by
$$(f + g)(x) := f(x) + g(x)$$
and scalar multiplication given by
$$(c \cdot f)(x) := c \cdot f(x).$$

Problem 2. Consider the $\mathbb{R}$-vector space $V = \mathbb{R}^3$ and the following subspaces

$$U_1 = \{(x_1, x_2, x_3) \in V : x_3 = 0\}, \quad U_2 = \{(x_1, x_2, x_3) \in V : x_2+3x_1 = 0, 4x_3-4x_2-12x_1 = 0\}$$
$$U_3 = \{(x_1, x_2, x_3) \in V : x_1 + x_2 = 0, 2x_2 - x_3 = 0\}.$$

Each item is worth 5 points. Solve the following parts:

(1) Describe the sums $U_1 + U_2$, $U_2 + U_3$ and $U_1 + U_3$.

(2) Describe the intersections $U_1 \cap U_2$, $U_2 \cap U_3$ and $U_1 \cap U_3$.

(3) Show that $V = U_1 \oplus U_3$ is the direct sum of U_1 and U_3.

(4) Write the vector $v = (5, -2, 1) \in V$ as $v = u_1 + u_3$, where $u_1 \in U_1$ and $u_3 \in U_3$. (By (3), this decomposition must be unique.)

(5) Find a vector subspace $W \subseteq V$ such that $V = W \oplus U_2$.

Problem 3. From the textbook. Solve the Proof-Writing Exercises (2), (3) and (4) in Page 47 (End of Chapter 4). The first two count 8 points and the last one 9 points.

Problem 4. **Prove**, with an argument, or **disprove**, with a counter-example, each of the statements sentences below. Each item is worth 5 points.

(1) Let $V = \mathbb{R}^4$ consider $U = \{(x_1, x_2, x_3, x_4) \in V : x_1+x_2 = 0, 2x_2-x_3 = 1\} \subseteq V$. Then U is a vector subspace.

(2) Let $V = \mathbb{R}[x]$ and consider $U = \{p(x) \in V : p(5) = 0 \text{ and } p(-7) = 0\} \subseteq V$. Then U is a vector subspace.

(3) Let $V = \mathbb{R}^5$ and consider the subspaces

$$U_1 = \{x_1 + x_2 - 4x_5 = 0, 2x_2 - 3x_3 + 8x_4 = 0\},$$
$$U_2 = \{5x_2 - 7x_3 + 4x_5 = 0, x_1 + 7x_2 + x_4 + x_5 = 0, x_5 + x_1 = 0\}.$$

Then $V = U_1 \oplus U_2$.

(4) Let $V = \mathbb{R}^4$, then the intersection $U_1 \cap U_2$ of the two planes

$$U_1 = \{x_1 - x_2 + x_4 = 0, 7x_1 + x_3 - 5x_4 = 0\},$$
$$U_1 = \{2x_1 + x_3 + 10x_4 = 0, x_2 + 4x_3 - 15x_4 = 0\}$$

is a line.

(5) Let $V = \mathbb{R}[x]$ and consider the subspaces

$$U_1 = \{p(x) \in V : p(0) = 0\},$$
$$U_1 = \{p(x) \in V : p(1) = 0\}.$$

Then $V = U_1 \oplus U_2$.