MAT 67: PROBLEM SET 2

DUE TO FRIDAY APR 19 2024

Abstract. This problem set corresponds to the second week of the course MAT-67 Spring 2024. Solutions were typed by TA Scroggin, please contact *tmscroggin – at – ucdavis.edu* for any comments.

Purpose: The goal of this assignment is to acquire the necessary skills to work with vector spaces. These were discussed during the second week of the course and are covered in Chapter 4 of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. Decide whether each of the following sets are $\mathbb{R}$-vector spaces and prove (or disprove) accordingly. Each item is worth 5 points:

(1) The set $\mathbb{R}^n$ with sum and scalar multiplications:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n).$$
$$c \cdot (x_1, \ldots, x_n) := (c \cdot x_1, \ldots, c \cdot x_n), \quad \forall c \in \mathbb{R}.$$

(2) Fix a natural number $n \in \mathbb{N}$. The set

$$P_{\leq n} := \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \quad (a_0, \ldots, a_n) \in \mathbb{R}^n\}$$

of polynomials in one variable x of degree at most n with sum

$$(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n) :=$$
$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n$$

and scalar multiplication

$$c \cdot (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) := ca_0 + ca_1 x + ca_2 x^2 + \ldots + ca_n x^n, \quad \forall c \in \mathbb{R}.$$

(3) Fix a natural number $n \in \mathbb{N}$. The set

$$P_n := \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : \quad (a_0, \ldots, a_n) \in \mathbb{R}^n, \quad a_n \neq 0\}$$

of polynomials in one variable x of degree exactly n with sum

$$(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) + (b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n) :=$$
$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots + (a_n + b_n)x^n$$

and scalar multiplication

$$c \cdot (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) := ca_0 + ca_1 x + ca_2 x^2 + \ldots + ca_n x^n, \quad \forall c \in \mathbb{R}.$$

(4) The set $\mathbb{Q} \subseteq \mathbb{R}$ of rational numbers with sum and scalar multiplications:

$$q_1 + q_2 := q_1 + q_2, \text{ the usual sum of rational numbers}$$
$$c \cdot q := c \cdot q, \text{ the usual product of a rational number } q \text{ by a real number } c$$

(5) The set $C(\mathbb{R}, \mathbb{R}) := \{f : \mathbb{R} \longrightarrow \mathbb{R} \text{ such that } f \text{ is a map}\}$ of maps from $\mathbb{R}$ to $\mathbb{R}$, with sum given by

$$(f + g)(x) := f(x) + g(x)$$

and scalar multiplication given by

$$(c \cdot f)(x) := c \cdot f(x).$$

Solution.

(1) $\boxed{Yes}$, this is a vector space.
We check that this space satisfies the vector space conditions. Let $x, y, z \in \mathbb{R}$ where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$, and let $a, b \in \mathbb{R}$. We rely heavily on the field properties of $\mathbb{R}$.

(a) Commutativity: We want to show that $x + y = y + x$ for all $x, y \in \mathbb{R}^n$.

$$\begin{aligned} x + y &= (x_1, \ldots, x_n) + (y_1, \ldots, y_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) \\ &= (y_1 + x_1, \ldots, y_n + x_n) && \text{(Commutativity of } \mathbb{R}\text{)} \\ &= (y_1, \ldots, y_n) + (x_1, \ldots, x_n) \\ &= y + x. \end{aligned}$$

(b) Associativity: We want to show that for all $x, y, z \in \mathbb{R}^n$ that $(x + y) + z = x + (y + z)$.

$$\begin{aligned} (x + y) + z &= [(x_1, \ldots, x_n) + (y_1, \ldots, y_n)] + (z_1, \ldots, z_n) \\ &= (x_1 + y_1, \ldots, x_n + y_n) + (z_1, \ldots, z_n) \\ &= ((x_1 + y_1) + z_1, \ldots, (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), \ldots, x_n + (y_n + z_n)) && \text{(Associativity of } \mathbb{R}\text{)} \\ &= (x_1, \ldots, x_n) + (y_1 + z_1, \ldots, y_n + z_n) \\ &= (x_1, \ldots, x_n) + [(y_1, \ldots, y_n) + (z_1, \ldots, z_n)] \\ &= x + (y + z). \end{aligned}$$

(c) Additive identity: $0 = (0, \ldots, 0) \in \mathbb{R}^n$ where

$$0 + x = (0, \ldots, 0) + (x_1, \ldots, x_n) = (0 + x_1, \ldots, 0 + x_n) = (x_1, \ldots, x_n) = x.$$

(d) Multiplicative identity: $1 \in \mathbb{R}$ and

$$1 \cdot x = 1 \cdot (x_1, \ldots, x_n) = (1 \cdot x_1, \ldots, 1 \cdot x_n) = (x_1, \ldots, x_n) = x.$$

(e) Additive inverses: If $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, then we have $-x = (-x_1, \ldots, -x_n) \in \mathbb{R}^n$ where

$$\begin{aligned} x + (-x) &= (x_1, \ldots, x_n) + (-x_1, \ldots, -x_n) \\ &= (x_1 - x_1, \ldots, x_n - x_n) = (0, \ldots, 0) = 0. \end{aligned}$$

(f) Distributivity: We want to show that $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$.

$$\begin{aligned}
a \cdot (x+y) &= a \cdot [(x_1, \dots, x_n) + (y_1, \dots, y_n)] \\
&= a \cdot (x_1 + y_1, \dots, x_n + y_n) \\
&= (a(x_1 + y_1), \dots, a(x_n + y_n)) \\
&= (ax_1 + ay_1, \dots, ax_n + ay_n) \\
&= (ax_1, \dots, ax_n) + (ay_1, \dots, ay_n) \\
&= a \cdot (x_1, \dots, x_n) + a \cdot (y_1, \dots, y_n) \\
&= a \cdot x + a \cdot y \\
(a+b) \cdot x &= (a+b) \cdot (x_1, \dots, x_n) \\
&= ((a+b) \cdot x_1, \dots, (a+b) \cdot x_n) \\
&= (a \cdot x_1 + b \cdot x_1, \dots, a \cdot x_n + b \cdot x_n) \\
&= (a \cdot x_1, \dots, a \cdot x_n) + (b \cdot x_1, \dots, b \cdot x_n) \\
&= a \cdot (x_1, \dots, x_n) + b \cdot (x_1, \dots, x_n) \\
&= a \cdot x + b \cdot x.
\end{aligned}$$

Since the space satisfies all the vector space criterion, it is therefore, a vector space.

(2) $\boxed{Yes}$, this is a vector space.

We show this space satisfies the vector space conditions. Let $a_0 + a_1 x + \cdots + a_n x^n,\ b_0 + b_1 x + \cdots + b_n x^n,\ c_0 + c_1 x + \cdots + c_n x^n \in P_{\leq n}$ and $\alpha, \beta \in \mathbb{R}$.

(a) Commutativity:

$$\begin{aligned}
(a_0 + a_1 x + \dots a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n) &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \\
&= (b_0 + a_0) + (b_1 + a_1)x + \dots (b_n + a_n)x^n \\
&= (b_0 + b_1 x + \cdots + b_n x^n) + (a_0 + a_1 x + \dots a_n x^n).
\end{aligned}$$

(b) Associativity:

$$\begin{aligned}
&[(a_0 + a_1 x + \cdots + a_n x^n) + (b_0 + b_1 x + \cdots + b_n x^n)] + (c_0 + c_1 x + \cdots + c_n x^n) \\
&= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n + (c_0 + c_1 x + \cdots + c_n x^n) \\
&= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)x + \cdots + (a_n + b_n + c_n)x^n \\
&= (a_0 + a_1 x + \cdots + a_n x^n) + ((b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n) \\
&= (a_0 + a_1 x + \cdots + a_n x^n) + [(b_0 + b_1 x + \cdots + b_n x^n) + (c_0 + c_1 x + \cdots + c_n x^n)].
\end{aligned}$$

(c) Additive identity: $0 \in P_{\leq n}$ and

$$(a_0 + a_1 x + \cdots + a_n x^n) + 0 = a_0 + a_1 x + \cdots + a_n x^n.$$

(d) Multiplicative identity: $1 \in \mathbb{R}$ and

$$1 \cdot (a_0 + a_1 x + \cdots + a_n x^n) = (1 \cdot a_0) + (1 \cdot a_1)x + \cdots + (1 \cdot a_n)x^n = a_0 + a_1 x + \cdots + a_n x^n.$$

(e) Additive inverse: If $a_0+a_1x+\dots a_nx^n \in P_{\leq n}$, then $-a_0-a_1x-\dots-a_nx^n \in P_{\leq n}$ where

$$\begin{aligned}&(a_0+a_1x+\dots a_nx^n)+(-a_0-a_1x-\dots-a_nx^n)\\&=(a_0-a_0)+(a_1-a_1)x+\dots+(a_n-a_n)x^n\\&=0+0x+\dots+0x^n=0.\end{aligned}$$

(f) Distributivity:

$$\begin{aligned}&\alpha\cdot[(a_0+a_1x+\dots+a_nx^n)+(b_0+b_1x+\dots+b_nx^n)]\\&=\alpha\cdot[(a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n]\\&=\alpha\cdot(a_0+b_0)+\alpha\cdot(a_1+b_1)x+\dots+\alpha\cdot(a_n+b_n)x^n\\&=(\alpha\cdot a_0+\alpha\cdot b_0)+(\alpha\cdot a_1+\alpha\cdot a_1)x+\dots+(\alpha\cdot a_n+\alpha\cdot b_n)x^n\\&=(\alpha\cdot a_0+\alpha\cdot a_1x+\dots+\alpha\cdot a_nx^n)+(\alpha\cdot b_0+\alpha\cdot b_1x+\dots+\alpha\cdot b_nx^n)\\&=\alpha\cdot(a_0+a_1x+\dots+a_nx^n)+\alpha\cdot(b_0+b_1x+\dots+b_nx^n)\end{aligned}$$

$$\begin{aligned}&(\alpha+\beta)\cdot(a_0+a_1x+\dots+a_nx^n)=(\alpha+\beta)\cdot a_0+(\alpha+\beta)\cdot a_1x+\dots+(\alpha+\beta)\cdot a_nx^n\\&=(\alpha\cdot a_0+\beta\cdot a_0)+(\alpha\cdot a_1+\beta\cdot a_1)x+\dots+(\alpha\cdot a_n+\beta\cdot a_n)x^n\\&=(\alpha\cdot a_0+\alpha\cdot a_1x+\dots+\alpha\cdot a_nx^n)+(\beta\cdot a_0+\beta\cdot a_1x+\dots+\beta\cdot a_nx^n)\\&=\alpha\cdot(a_0+a_1x+\dots a_nx^n)+\beta\cdot(a_0+a_1x+\dots a_nx^n)\end{aligned}$$

By satisfying the above conditions, $P_{\leq n}$ is a vector space.

(3) $\boxed{No}$, this is not a vector space.
We provide a counterexample. Let $n=2$, then $1+x+x^2,\ 1+x-x^2 \in P_2$; however, we have that

$$(1+x+x^2)+(1+x-x^2)=2+2x\notin P_2.$$

Therefore, since the additive operation is not closed, i.e., the sum of two elements in the space is not in the space, then P_n cannot be a vector space.

(4) $\boxed{No}$, this is not an $\mathbb{R}$-vector space.
We provide a counterexample. We know that $1\in\mathbb{Q}$ and that $\pi\in\mathbb{R}$, if $\mathbb{Q}$ is an $\mathbb{R}$ vector space then we should have that $\pi\cdot 1=\pi\in\mathbb{Q}$. However, π is an irrational number and not an element of $\mathbb{Q}$ which is a contradiction.

(5) The set $C(\mathbb{R},\mathbb{R}) := \{f:\mathbb{R}\longrightarrow\mathbb{R}$ such that f is a map$\}$ of maps from $\mathbb{R}$ to $\mathbb{R}$, with sum given by

$$(f+g)(x):=f(x)+g(x)$$

and scalar multiplication given by

$$(c\cdot f)(x):=c\cdot f(x).$$

□

Problem 2. Consider the $\mathbb{R}$-vector space $V=\mathbb{R}^3$ and the following subspaces

$$U_1=\{(x_1,x_2,x_3)\in V : x_3=0\},\quad U_2=\{(x_1,x_2,x_3)\in V : x_2+3x_1=0, 4x_3-4x_2-12x_1=0\}$$

$$U_3=\{(x_1,x_2,x_3)\in V : x_1+x_2=0, 2x_2-x_3=0\}.$$

Each item is worth 5 points. Solve the following parts:

(1) Describe the sums $U_1 + U_2$, $U_2 + U_3$ and $U_1 + U_3$.

(2) Describe the intersections $U_1 \cap U_2$, $U_2 \cap U_3$ and $U_1 \cap U_3$.

(3) Show that $V = U_1 \oplus U_3$ is the direct sum of U_1 and U_3.

(4) Write the vector $v = (5, -2, 1) \in V$ as $v = u_1 + u_3$, where $u_1 \in U_1$ and $u_3 \in U_3$. (By (3), this decomposition must be unique.)

(5) Find a vector subspace $W \subseteq V$ such that $V = W \oplus U_2$.

Solution.

(1) We note that U_1 is the xy-plane in $\mathbb{R}^3$ and therefore,
$$U_1 := \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\}.$$
Now, we solve the system of equations which describe U_2 and U_3 so we may write the subsets in terms of the vectors which describe the subspace.

For U_2, in the first equation we solve for x_2 and get $x_2 = -3x_1$. We plug the value of x_2 into the second equation to solve for x_3, we get
$$\begin{aligned} 4x_3 - 4(-3x_1) - 12x_1 &= 0 \\ 4x_3 &= 0 \\ x_3 &= 0. \end{aligned}$$
Therefore, the subspace U_2 is defined
$$U_2 := \{(x_1, -3x_1, 0) \in \mathbb{R}^3 : x_1 \in \mathbb{R}\}$$
which is the line $y = -3x$ in the xy-plane passing through the vector $(1, -3, 0)$.

As for U_3, we solve for x_1 in the first equation and find that $x_2 = -x_1$. We plug the value of x_2 into the second equation to find x_3
$$\begin{aligned} 2(-x_1) - x_3 &= 0 \\ -2x_1 &= x_3. \end{aligned}$$
Therefore, the subspace U_3 is defined
$$U_3 := \{(x_1, -x_1, -2x_1) \in \mathbb{R}^3 : x_1 \in \mathbb{R}\}$$
which is a line in $\mathbb{R}^3$ passing through the vector $(1, -1, -2)$.

Now, we may describe the sums:
$$\begin{aligned} U_1 + U_2 &= \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\} \\ &= \{(x_1, x_2, 0) + (x_1', -3x_1, 0) : x_1, x_2, x_1' \in \mathbb{R}\} \\ &= \{(x_1 + x_1', x_2 - 3x_1, 0) : x_1, x_2, x_1' \in \mathbb{R}\} \end{aligned}$$
Since $U_2 \subseteq U_1$ then $U_1 + U_2 = U_1$

$$\begin{aligned} U_1 + U_3 &= \{u_1 + u_3 : u_1 \in U_1, u_3 \in U_3\} \\ &= \{(x_1, x_2, 0) + (x_1', -x_1', -2x_1') : x_1, x_2, x_1' \in \mathbb{R}\} \\ &= \{(x_1 + x_1', x_2 - x_1', -2x_1') : x_1, x_2, x_1' \in \mathbb{R}\} \end{aligned}$$

We note that $\boxed{U_1 + U_3 = \mathbb{R}^3}$, since we may write any $(x, y, z) \in \mathbb{R}^3$ as a linear combination of vectors in $U_1 + U_3$.

$$\begin{cases} x_1 + x_1' = x \\ x_2 - x_1' = y \\ -2x_1' = z \end{cases}$$

Where $x_1 = x + \frac{1}{2}z$, $x_2 = y - \frac{1}{2}z$, $x_1' = -\frac{1}{2}z$.

$$\begin{aligned} U_2 + U_3 &= \{u_2 + u_3 : u_2 \in U_2, u_3 \in U_3\} \\ &= \{(x_1, -3x_1, 0) + (x_1', -x_1', -2x_1') : x_1, x_1' \in \mathbb{R}\} \\ &= \{(x_1 + x_1', -3x_1 - x_1', -2x_1') : x_1, x_1' \in \mathbb{R}\} \end{aligned}$$

To completely describe $U_2 + U_3$ we must determine the equation for the plane in $\mathbb{R}^3$, for this we'll need some machinery from MAT 21D. we see that the vector $u_2 = (1, -3, 0) \in U_2$ and $u_3 = (1, -1, -2) \in U_3$. We use these vectors to compute the normal vector to plane, then use the fact that the vector $(0, 0, 0) \in U_2 + U_3$.

$$\vec{n} = u_2 \times u_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 6\hat{i} + 2\hat{j} + 2\hat{k}$$

Now, we determine the equation for the tangent plane using

$$(x - x_0, y - y_0, z - z_0) \cdot \vec{n} = 0.$$

Here, we let $(x_0, y_0, z_0) = (0, 0, 0)$.

$$\begin{aligned} (x, y, z) \cdot (6, 2, 2) &= 0 \\ 6x + 2y + 2z &= 0 \end{aligned}$$

Therefore, $U_2 + U_3$ is given by the plane $\boxed{\text{6x+2y+2z=0.}}$

(2) From part (1), we determined that $U_2 \subseteq U_1$, therefore, $U_1 \cap U_2 = \boxed{U_2.}$
For $U_1 \cap U_3$, we determine when $(x_1, x_2, 0) = (x_1', -x_1', -2x_1')$. Since $0 = -2x_1'$, then $x_1' = 0$ forcing $x_2 = -x_1' = 0$ and $x_1 = x_1' = 0$. Therefore, $U_1 \cap U_3 = \boxed{\{0\}.}$
For $U_2 \cap U_3$, we determine when $(x_1, -3x_1, 0) = (x_1', -x_1', -2x_1')$. Since $0 = -2x_1'$, then $x_1' = 0$ and $x_1 = 0$. Hence, $U_2 \cap U_3 = \boxed{\{0\}.}$

(3) Since $U_1 + U_3 = \mathbb{R}^3$ by part (1) and that $U_1 \cap U_3 = \{0\}$ by part (2) we have that $U_1 \oplus U_3 = \mathbb{R}^3 = V$.

(4) Using the equations found in part (1) to show that $U_1 + U_3 = \mathbb{R}^3$, we have that

$$\begin{aligned} x_1 &= x + \frac{1}{2}z = 5 + \frac{1}{2}(1) = \frac{11}{2} \\ x_2 &= y - \frac{1}{2}z = -2 - \frac{1}{2}(1) = -\frac{5}{2} \\ x_1' &= -\frac{1}{2}z = -\frac{1}{2}(1) = -\frac{1}{2}. \end{aligned}$$

Therefore,

$$v = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{11}{2} \\ -\frac{5}{2} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix} 11 \\ -5 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.}$$

(5) Since U_2 is line in $\mathbb{R}^3$ we can consider the vector $u_2 = (1, -3, 0)$ as the normal vector to the plane which describes W, where $(0, 0, 0) = U_2 \cap W$. Therefore, the plane is described as

$$(x, y, z) \cdot (1, -3, 0) = x - 3y = 0.$$

As a set we have that

$$\boxed{W = \{(3x_2, x_2, x_3) \in \mathbb{R}^3 : x_1, x_3 \in \mathbb{R}\}.}$$

□

Problem 3. From the textbook. Solve the Proof-Writing Exercises (2), (3) and (4) in Page 47 (End of Chapter 4). The first two count 8 points and the last one 9 points.

Solution.

(1) *Exercise 4.2*: Let V be a vector space over $\mathbf{F}$, suppose that W_1 and W_2 are subspaces of V. Prove that their intersection $W_1 \cap W_2$ is also a subspace of V.

Proof. Given that W_1, W_2 are both subspaces of V then both W_1 and W_2 contain 0; therefore, $0 \in W_1 \cap W_2$.

Now, we want to show that if $v_1, v_2 \in W_1 \cap W_2$ then $v_1 + v_2 \in W_1 \cap W_2$. If $v_1, v_2 \in W_1 \cap W_2$ then v_1, v_2 are in both W_1 and W_2. By the subspace properties, we know that $v_1 + v_2 \in W_1$ and $v_1 + v_2 \in W_2$; hence, $v_1 + v_2 \in W_1 \cap W_2$.

Finally, we want to show that if $v \in W_1 \cap W_2$ and $c \in \mathbf{F}$, then $c \cdot v \in W_1 \cap W_2$. Given that W_1 is a subspace then $c \cdot v \in W_1$ and by the same reasoning $c \cdot v \in W_2$; hence, $c \cdot v \in W_1 \cap W_2$. Thereby proving that if $W_1, W_2 \subseteq V$ are subspaces, then the intersection $W_1 \cap W_2$ is also a subspace of V.

□

(2) *Exercise 4.3*: Prove or give a counterexample to the following claim:
Claim: Let V be a vector space over $\mathbf{F}$, and suppose that W_1, W_2, W_3 are subspaces of V such that $W_1 + W_3 = W_2 + W_3$. Then $W_1 = W_2$.

This statement is $\boxed{\textit{false.}}$

Let W_3 be the xy-plane and define

$$W_1 := \{(x_1, 2x_1) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\},$$
$$W_2 := \{(x_1, -3x_1) \in \mathbb{R}^2 : x_1 \in \mathbb{R}\}.$$

Since $W_1 \subseteq W_3$ and $W_2 \subseteq W_3$ then $W_1 + W_3 = W_3$ and $W_2 + W_3 = W_3$; however, W_1 and W_2 are both distinct lines in the plane, they are not equivalent as vector subspaces.

(3) *Exercise 4.4*: Prove or give a counterexample to the following claim:
Claim: Let V be a vector space over $\mathbf{F}$, and suppose that W_1, W_2, W_3 are subspaces of V such that $W_1 \oplus W_3 = W_2 \oplus W_3$. Then $W_1 = W_2$.

This statement is $\boxed{false.}$

Let W_3 be the xy-plane and define

$$W_1 := \{(0, 0, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\},$$
$$W_2 := \{(0, x_3, x_3) \in \mathbb{R}^3 : x_3 \in \mathbb{R}\}.$$

Then $W_1 \oplus W_3 = \mathbb{R}^3$ since $W_1 + W_3 = \mathbb{R}^3$ and $W_1 \cap W_3 = \{0\}$ and $W_2 \oplus W_3 = \mathbb{R}^3$ since $W_2 + W_3 = \mathbb{R}^3$ and $W_2 \cap W_3 = \{0\}$. However, $W_1 \neq W_2$.

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Problem 4. **Prove**, with an argument, or **disprove**, with a counter-example, each of the statements sentences below. Each item is worth 5 points.

(1) Let $V = \mathbb{R}^4$ consider $U = \{(x_1, x_2, x_3, x_4) \in V : x_1 + x_2 = 0, 2x_2 - x_3 = 1\} \subseteq V$. Then U is a vector subspace.

(2) Let $V = \mathbb{R}[x]$ and consider $U = \{p(x) \in V : p(5) = 0 \text{ and } p(-7) = 0\} \subseteq V$. Then U is a vector subspace.

(3) Let $V = \mathbb{R}^5$ and consider the subspaces

$$U_1 = \{x_1 + x_2 - 4x_5 = 0, 2x_2 - 3x_3 + 8x_4 = 0\},$$
$$U_2 = \{5x_2 - 7x_3 + 4x_5 = 0, x_1 + 7x_2 + x_4 + x_5 = 0, x_5 + x_1 = 0\}.$$

Then $V = U_1 \oplus U_2$.

(4) Let $V = \mathbb{R}^4$, then the intersection $U_1 \cap U_2$ of the two planes

$$U_1 = \{x_1 - x_2 + x_4 = 0, 7x_1 + x_3 - 5x_4 = 0\},$$
$$U_2 = \{2x_1 + x_3 + 10x_4 = 0, x_2 + 4x_3 - 15x_4 = 0\}$$

is a line.

(5) Let $V = \mathbb{R}[x]$ and consider the subspaces

$$U_1 = \{p(x) \in V : p(0) = 0\},$$
$$U_2 = \{p(x) \in V : p(1) = 0\}.$$

Then $V = U_1 \oplus U_2$.

Solution.

(1) This statement is $\boxed{false.}$

Using the first equation we find that $x_2 = -x_1$ which allows us to solve for $x_3 = -2x_1 - 1$. Therefore,

$$U = \{(x_1, -x_1, -2x_1 - 1) : x_1 \in \mathbb{R}\}.$$

Let $(x_1, -x_1, -2x_1 - 1), (x_1', -x_1', -2x_1' - 1) \in U$, then $(x_1, -x_1, -2x_1 - 1) + (x_1', -x_1', -2x_1' - 1)(x_1 + x_1', -(x_1 + x_1'), -2(x_1 + x_1') - 2) \notin U$, violating the vector addition property. We may have also shown that this subset is not closed under scalar multiplication.

(2) This statement is $\boxed{true.}$

First, we determine the form of elements in U. Since $p(5) = 0$ then all polynomials which satisfy this condition must be of the form $f(x) = f_1(x) \cdot (x - 5)$ where $f_1(x) \in \mathbb{R}[x]$, i.e., 5 is a root of the polynomial f since $f(5) = f_1(5) \cdot (5 - 5) = f_1(5) \cdot 0 = 0$. Note that $f_1(5)$ does not necessarily equal 0. Similarly, polynomials which satisfy the condition that $p(-7) = 0$ are of the form $g(x) = g_1(x) \cdot (x + 7)$ where $g_1(x) \in \mathbb{R}[x]$.

Since U is defined to be the set of all polynomials which both satisfy $p(5) = 0$ and $p(-7) = 0$, then polynomials in this set must be of the form $f(x)(x-5)(x+7)$ where $f(x) \in \mathbb{R}[x]$,i.e.,

$$U := \{f(x)(x - 5)(x + 7) : f(x) \in \mathbb{R}[x]\}.$$

Now, we want to show that U is a vector subspace, i.e., that $0 \in U$ and for all $f(x), g(x) \in U$ and $c \in \mathbb{R}$ we have that $f(x) + g(x) \in U$ and $c \cdot f(x)$. Clearly, $0 \in U$ since the constant 0 function evaluated at 5 and at -7 is the constant function 0. Now, let $f(x) = f_1(x)(x - 5)(x + 7)$ and $g(x) = g_1(x)(x-)(x + 7)$, then

$$\begin{aligned} f(x) + g(x) &= f_1(x)(x - 5)(x + 7) + g_1(x)(x - 5)(x + 7) \\ &= [f_1(x) + g_1(x)](x - 5)(x + 7) \end{aligned}$$

Since $f_(x) + g_1(x) \in \mathbb{R}[x]$, then $f(x) + g(x) \in U$. Finally, we check that $c \cdot f(x) \in U$,

$$c \cdot f(x) = c \cdot f_1(x)(x - 5)(x + 7) = (c \cdot f_(x))(x - 5)(x + 7).$$

Since $c \cdot f_1(x) \in \mathbb{R}[x]$, then $c \cdot f(x) \in U$, thus proving that U satisfies the vector subspace conditions.

(3) This statement is $\boxed{true.}$

We begin by simplifying the sets U_1 and U_2 into its vector representation.

For U_1, we use the first equation to solve $x_2 = -x_1 + 4x_5$. Using the second equation we find that

$$\begin{aligned} 2(-x_1 + 4x_5) - 3x_3 + 8x_4 &= 0 \\ -2x_1 + 8x_5 - 3x_3 + 8x_4 &= 0 \\ x_4 &= \frac{1}{4}x_1 + \frac{3}{8}x_3 - x_5. \end{aligned}$$

We let x_1, x_3, x_5 be free variables which allows us to express U_1 as

$$U_1 := \left\{ \left(x_1, -x_1 + 4x_5, x_3, \frac{1}{4}x_1 + \frac{3}{8}x_3 - x_5, x_5 \right) : x_1, x_3, x_5 \in \mathbb{R} \right\}$$

We note that U_1 is of real dimension 3.

For U_2, we start by finding $x_1 = -x_5$ from the third equation. Using this fact in the second equation, we obtain

$$\begin{aligned} 7x_2 + x_4 &= 0 \\ x_2 &= -\frac{1}{7}x_4. \end{aligned}$$

From the first equation we solve for x_2,

$$x_2 = \frac{7}{5}x_3 - \frac{4}{5}x_5.$$

Using the previous two equations allows us to solve for x_4

$$\begin{aligned} -\frac{7}{4}x_4 &= \frac{7}{5}x_3 - \frac{4}{5}x_5 \\ x_4 &= -\frac{49}{5}x_3 + \frac{28}{5}x_5 \end{aligned}$$

Therefore, the set U_2 is defined

$$U_2 := \left\{ \left(-x_5, \frac{7}{5}x_3 - \frac{4}{5}x_5, x_3, -\frac{49}{5}x_3 + \frac{28}{5}x_5, x_5 \right) : x_1, x_5 \in \mathbb{R} \right\}.$$

We note that U_2 is of real dimension 2.

Now, we determine the intersection $U_1 \cap U_2$, i.e., we determine when

$$\left(x_1, -x_1 + 4x_5, x_3, \frac{1}{4}x_1 + \frac{3}{8}x_3 - x_5, x_5 \right) = \left(-x_5', \frac{7}{5}x_3' - \frac{4}{5}x_5', x_3', -\frac{49}{5}x_3' + \frac{28}{5}x_5', x_5' \right).$$

We see that $x_3 = x_3'$, $x_5 = x_5'$ and $x_1 = x_5' = x_5$. From the second coordinate

$$\begin{aligned} -x_1 + 4x_5 &= \frac{7}{5}x_3 - \frac{4}{5}x_5 \\ -(-x_5) + 4x_5 &= \frac{7}{5}x_3 - \frac{4}{5}x_5 \\ 5x_5 + \frac{4}{5}x_5 &= \frac{7}{5}x_3 \\ \frac{29}{5}x_5 &= \frac{7}{5}x_3 \\ \frac{29}{7}x_5 &= x_3. \end{aligned}$$

From the fourth coordinate

$$\begin{aligned}
\frac{1}{4}x_1 + \frac{3}{8}x_3 - x_5 &= -\frac{49}{5}x_3 + \frac{28}{5}x_5 \\
\frac{1}{4}(-x_5) + \frac{3}{8}\left(\frac{29}{7}x_5\right) - x_5 &= -\frac{49}{5}\left(\frac{29}{7}x_5\right) + \frac{28}{5}x_5 \\
\left(-\frac{1}{4} + \frac{3}{8}\cdot\frac{29}{7} - 1\right)x_5 &= \left(-\frac{49}{5}\cdot\frac{29}{7} + \frac{28}{5}\right)x_5 \\
x_5 &= 0
\end{aligned}$$

Since $x_5 = 0$ then $U_1 \cap U_2 = (0,0,0,0,0)$.

Given that the two subspaces intersect only at the origin, i.e., these two subspaces are linearly independent, and that the dimensions sum to the total dimension of the space, i.e., $\dim(U_1) + \dim(U_2) = 2 + 3 = 5 = \dim(\mathbb{R}^5)$, then $V = U_1 \oplus U_2$.

(4) This statement is $\boxed{false.}$ These two planes intersect at the point $(0,0,0,0)$.

To see this, we first simplify the sets U_1 and U_2 into its vector representation.

For U_1, we first solve for $x_1 = x_2 - x_4$ from the first equation. From the second equation we see that

$$\begin{aligned}
7(x_2 - x_4) + x_3 - 5x_4 &= 0 \\
7x_2 - 7x_4 + x_3 - 5x_4 &= 0 \\
7x_2 + x_3 - 12x_4 &= 0 \\
x_3 &= -7x_2 + 12x_4
\end{aligned}$$

We let x_2, x_4 be free variables and find that

$$U_1 := \{(x_2 - x_4, x_2, -7x_2 + 12x_4, x_4) : x_2, x_4 \in \mathbb{R}\}.$$

For U_2, from the second equation we solve for x_3 and find that

$$x_3 = \frac{1}{4}x_2 + \frac{15}{4}x_4.$$

Using the first equation we find that $x_1 = -\frac{1}{2}x_3 - 5x_4$. Now, using the equation for x_3 from the second equation we can further solve for x_1 in terms of x_2, x_4 which are free variables in the set U_1.

$$\begin{aligned}
x_1 &= -\frac{1}{2}\left(\frac{1}{4}x_2 + \frac{15}{4}x_4\right) - 5x_4 \\
&= -\frac{1}{8}x_2 - \frac{15}{8}x_4 - 5x_4 \\
&= -\frac{1}{8}x_2 - \frac{55}{8}x_4.
\end{aligned}$$

This allows us to write

$$U_2 := \left\{\left(-\frac{1}{8}x_2 - \frac{55}{8}x_4, x_2, \frac{1}{4}x_2 + \frac{15}{4}x_4, x_4\right) : x_2, x_4 \in \mathbb{R}\right\}.$$

Now, we can see that the two equations

$$x_2 - x_4 = -\frac{1}{8}x_2 - \frac{55}{8}x_4$$

$$-7x_2 + 12x_4 = \frac{1}{4}x_2 + \frac{15}{4}x_4$$

may only be satisfied if both $x_2, x_4 = 0$. Therefore, the only point of intersection is $(0,0,0,0)$.

(5) This statement is $\boxed{false.}$

We provide a counterexample to the fact that $U_1 \cap U_2 = \{0\}$. Let $p(x) = x - x^2$. Then $p(0) = 0 - 0^2 = 0$ and $p(1) = 1 - 1^2 = 1 - 1 = 0$. Therefore, the intersection $U_1 \cap U_2 \neq \{0\}$ violating the direct sum requirements.

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