MAT 67: PROBLEM SET 3

DUE TO FRIDAY MAY 10 2024

Abstract. This problem set corresponds to the fifth week of the course MAT-67 Spring 2024. It is due Friday May 10 at 9:00am submitted via Gradescope.

Purpose: The goal of this assignment is to acquire the necessary skills to work with linear maps, bases and matrices. These were discussed during the fifth week of the course and are covered in Chapter 6 and Appendix A of the textbook.

Task: Solve Problems 1 through 4 below.

Instructions: It is perfectly good to consult with other students and collaborate when working on the problems. However, you should write the solutions on your own, using your own words and thought process. List any collaborators in the upper-left corner of the first page.

You are welcome to use the Office Hours offered by the Professor and the TA. Again, list any collaborators or contributors in your solutions. Make sure you are using your own thought process and words, even if an idea or solution came from elsewhere. (In particular, it might be wrong, so please make sure to think about it yourself.)

Grade: Each graded Problem is worth 25 points, the total grade of the Problem Set is the sum of the number of points. The maximum possible grade is 100 points.

Writing: Solutions should be presented in a balanced form, combining words and sentences which explain the line of reasoning, and also precise mathematical expressions, formulas and references justifying the steps you are taking are correct. If you are using theorems in lecture and in the textbook, make that reference clear. (E.g. specify name/number of the theorem and section of the book.)

Problem 1. Let $V = W = \mathbb{R}^3$ and let $f : V \to W$ be a linear map. Consider the basis $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$, where

$$\begin{gathered} v_1 = (1, 0, -2), \quad v_2 = (3, 4, 0), \quad v_3 = (1, -1, 2), \\ w_1 = (1, 0, 1), \quad w_2 = (1, 1, 1), \quad w_3 = (0, 0, 1). \end{gathered}$$

Suppose that $f(v_i) = w_i$ for $1 \leq i \leq 3$.

(1) Find the vectors $f(5, 3, 0)$, $f(5, 9, -2)$ and $f(1, 0, 0)$.

(2) Find the numbers $a_{ij} \in \mathbb{R}$ such that

$f(x_1, x_2, x_3) = (a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3).$

Problem 2. Consider $V = \mathbb{R}^3, W = \mathbb{R}^2, Z = \mathbb{R}^2$ and the following two linear maps $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$:

$$f(x_1, x_2, x_3) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 5 & -5 \\ -4 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad g(y_1, y_2, y_3) = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 6 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Each item is worth 5 points. Solve the following parts:

(1) Compute $f(1, 3, -1)$ and $g(2, 5, 0)$.

(2) Find a matrix expression for the composition $g \circ f : \mathbb{R}^3 \to \mathbb{R}^2$.

(3) Find bases for the nullspaces $\ker(f)$, $\ker(g)$ and $\ker(g \circ f)$.

(4) Find bases for the ranges $im(f)$, $im(g)$ and $im(g \circ f)$.

Problem 3. From the textbook. Solve the Exercises (1), (2) and (6) in Page 86 (End of Chapter 6). The first two count 8 points and the last one 9 points.

Problem 4. From the textbook. Solve the Proof-Writing Exercises (1), (2), (4) and (7) in Page 83 (End of Chapter 6). The first three count 7 points and the last one 4 points.