Counting and Recounting

The beauty of combinatorial proofs... the ultimate understanding of an identity!!!

Marta Sved

n! or n^k
"Counting numbers", aren't they!
Thou shalt not rush without compunction
To that transcendental function
To explain the situation
Why they are in that equation.
Thou shalt try to get a gleaning
Of that simple real meaning.
Thy mind shall turn pictorial
Seeing n!

A large collection of algebraic formulae and identities deals with a finite number of integers or rational numbers. While powerful techniques of algebra and analysis are available to establish such identities, there remains a driving compulsion to give "natural" interpretations to relations dealing with "natural" numbers. Such a compulsion produced an "elementary" proof of the prime number theorem, even though analytical methods had proved it much earlier with elegant efficiency. It is always a challenge to take out a combinatorial relation from its formula deep freeze and restore it to physical life: this procedure is the reversal of what we do in the usual mathematical problem solving course. Here are some nice examples.

Some Product Formulae

The combinatorial meanings of $n!$, $n^{(r)} = n(n-1)\dots(n-r+1)$, $\binom{n}{r} = n!/r!\,(n-r)!$, m^n are presented in every introduction to combinatorics, but treating combinations of these expressions, we let this combinatorial meaning retreat. It is easy enough to show that $(kn)!$ is more than $(k!)^n$ whenever k is more than 1, but the best way to show that $(k!)^n$ is in fact a proper divisor of $(kn)!$ is to think of $(kn)!/(k!)^n$ as the number of ways of dividing kn people into n sets. For $k = 4$, this is the number of ways to allocate n bridge tables to players.

From bridge to tennis: We can interpret the identities

$$\binom{2}{2}\binom{4}{2}\dots\binom{2n-2}{2}\binom{2n}{2}$$

$$= n!(2n-1)(2n-3)\dots5.3.1 = \frac{(2n)!}{2^n}$$

as the number of allocations of n tennis courts to $2n$ players.

On the left side, we assign 2 players out of $2n$ to the first court, 2 out of the remaining $2n-2$ to the second one, and so on.

In the middle, we begin with any of the players, choose his opponent out of $2n-1$, continue with any one of the remaining players, choosing his opponent out of $2n-3$ and continue in this fashion to obtain $(2n-1)(2n-3)\dots5.3.1$ pairings. Next we assign the courts to the n pairs in $n!$ ways.

On the right side, we simply line up the players and lead them in pairs to the courts. The lining up can be done in $(2n)!$ ways. Since we do not fix the allocation of the north and south ends of the courts, we divide this number by 2^n.

Now let's tackle something more sophisticated:

$$(-1)^n\binom{2n+1}{n} = 2^{2n+1}(2n+1)\binom{1/2}{n+1}$$

We first take it to the form

$$\binom{2n}{n}n! = 2^n.1.3.5\dots(2n-1)$$

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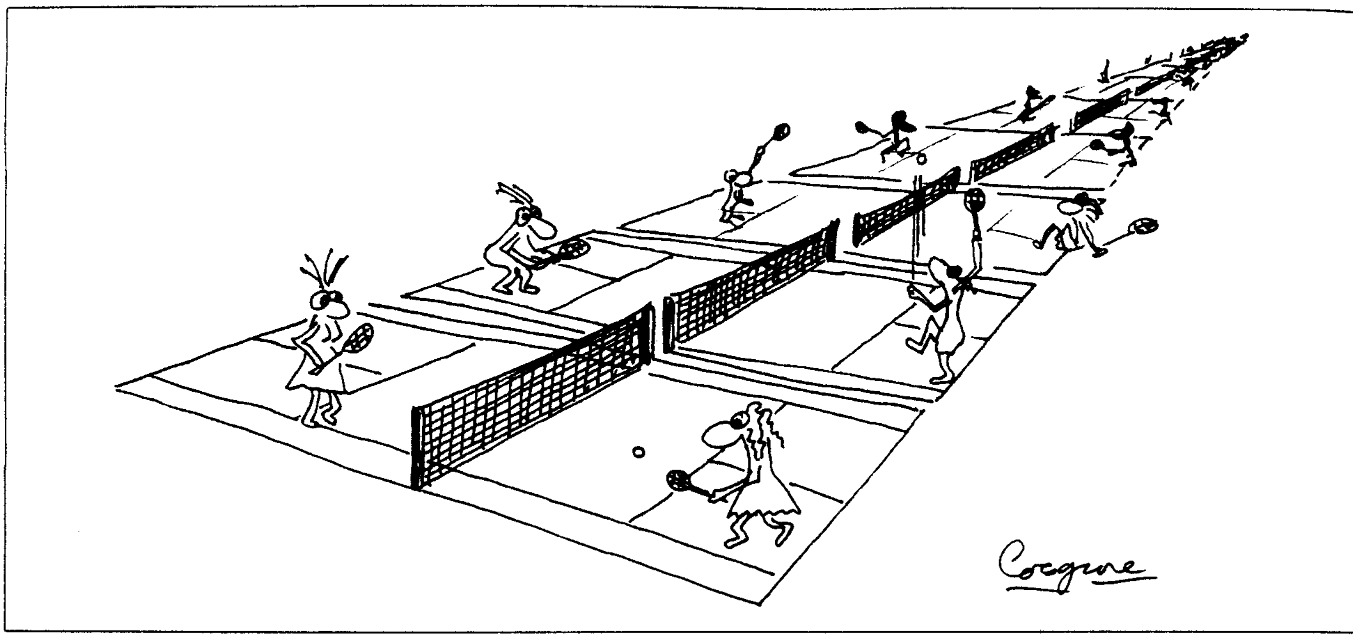


and then consider the outcome of the first round of our tennis tournament.

The right hand side of the identity shows that the first round was organized in $(2n-1)(2n-3)\ldots 3.1$ ways, (as in the previous identity), with 2 outcomes possible for each pair of opponents.

For the left hand side we notice that, whatever the pairings will be, there will be $\binom{2n}{n}$ ways in which the winners can emerge. For each set of winners there are $n!$ ways in which the losing opponents are distributed.

Chess instead of tennis? In this case the first round may end with some draws. By the same reasoning we obtain the rather formidable identity:

$$\sum_{k=0}^{n-1}\binom{2n}{2k}\binom{2k}{k}k!\prod_{j=1}^{n-k}(2j-1)+\binom{2n}{n}n!$$

$$= 3^n.1.3.5\ldots(2n-1)$$

where the k index of the summation means the number of decisive games.

This last identity, however, belongs to the class of combinatorial formulae sampled in the next paragraph.

2. Identities Involving Summations

Here are a few examples of the inexhaustible number of summation identities.

The basic identity involving binomial coefficients is the Pascal triangle relation

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k}.$$

The combinatorial interpretation of this is well known, so we begin with two of its immediate consequences:

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-2}{k-1}+\ldots+\binom{k-1}{k-1} \qquad \textbf{(1)}$$

$$\binom{n}{k}=\binom{n-r}{k}+\binom{r}{1}\binom{n-r}{k-r}+\ldots+\binom{r}{i}\binom{n-r}{k-i}$$

$$+\ldots+\binom{n-r}{k-r}\text{ for } r\leq k. \qquad \textbf{(2)}$$

Both these identities can be interpreted as selection procedures of k children out of a class of n.

For (1) we proceed by first ordering the class in alphabetical order: $A_1, A_2 \ldots A_n$ and calling the selected set in alphabetical order. A_1 can head $\binom{n-1}{k-1}$ sets. When these are exhausted, we consider the $\binom{n-2}{k-1}$ sets headed by A_2 and so on until we get finally to the single possible set headed by A_{n-k+1}.

In (2) we do the selections by paying special attention to the r girls in the mixed class of n children (at the risk of being called sexist). If we are sufficiently prejudiced not to want any of them on the team, we can make $\binom{n-r}{k}$ choices out of the $n-r$ boys in the class. If we admit one girl who can be selected in $\binom{r}{1}$ ways, we choose the boys in $\binom{n-r}{k-1}$ ways. Finally we give full preference to the girls, selecting all of them, and select $k-r$ boys in $\binom{n-r}{k-r}$ ways.

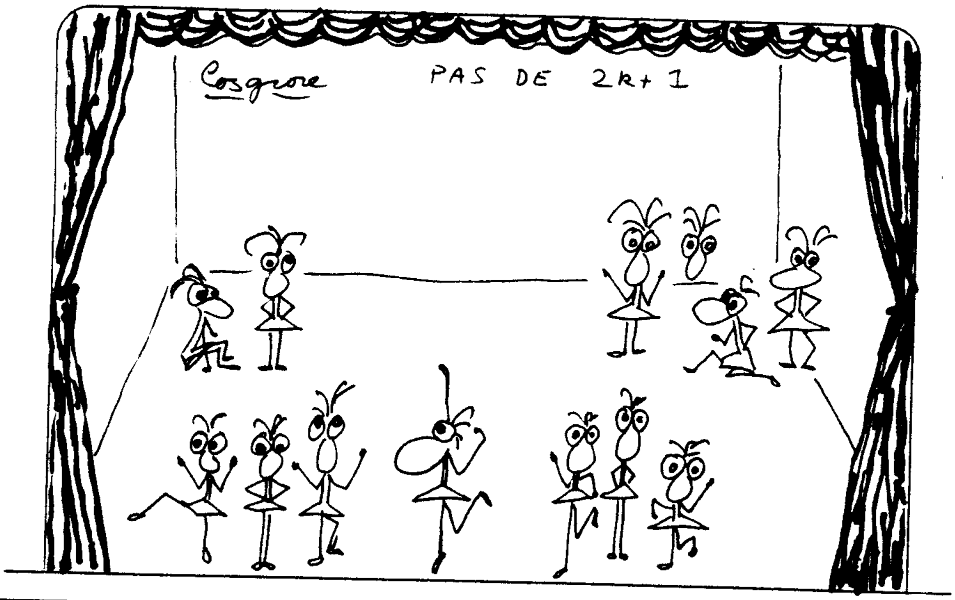
Identity (2) is perhaps better known in the more symmetrical form where the number of boys is denoted by n_1, the number of girls by n_2, obtaining

$$\binom{n_1+n_2}{k}=\sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i}$$

(Vandermonde convolution)

In particular for $n_1 = n_2 = k = n$ we obtain

$$\binom{2n}{n}=\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}=\sum_{i=0}^{n}\binom{n}{i}^2.$$



This last formula can be given a little twist to obtain

$$\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}.$$

Here's our story: Two schools, A and B, send a combined team for an interstate competition. The team must consist of n members, but the captain must come from school A. Assuming that each school provides a preselection of n students, we want to determine the numbers of possible final selections. The left hand side shows the arrangements when i members of the team come from school A, $n - i$ from B and then the captain is selected from amongst the i team-mates from A, where i runs from 1 to n. On the right hand side we do the selection by first choosing the captain from the A side and then choosing the remaining members.

We move on to the dancing class to obtain

$$\sum_{r=k}^{n-k}\binom{r}{k}\binom{n-r}{k} = \binom{n+1}{2k+1}.$$

This particular class has $n + 1$ children, but only $2k + 1$ are needed for a certain dance. The ballet master lines them up (keeping the better dancers near the centre), and then chooses the child to dance at the centre, k on her left and k on her right. If the centre dancer is the $(r + 1)$-st child, then $r \geqslant k$ and $n - r \geqslant k$; hence r runs from k to $n - k$.

A beautiful identity known as Riordan's identity,

$$n^n = \sum_{k=1}^{n}\binom{n-1}{k-1}n^{n-k}k!$$

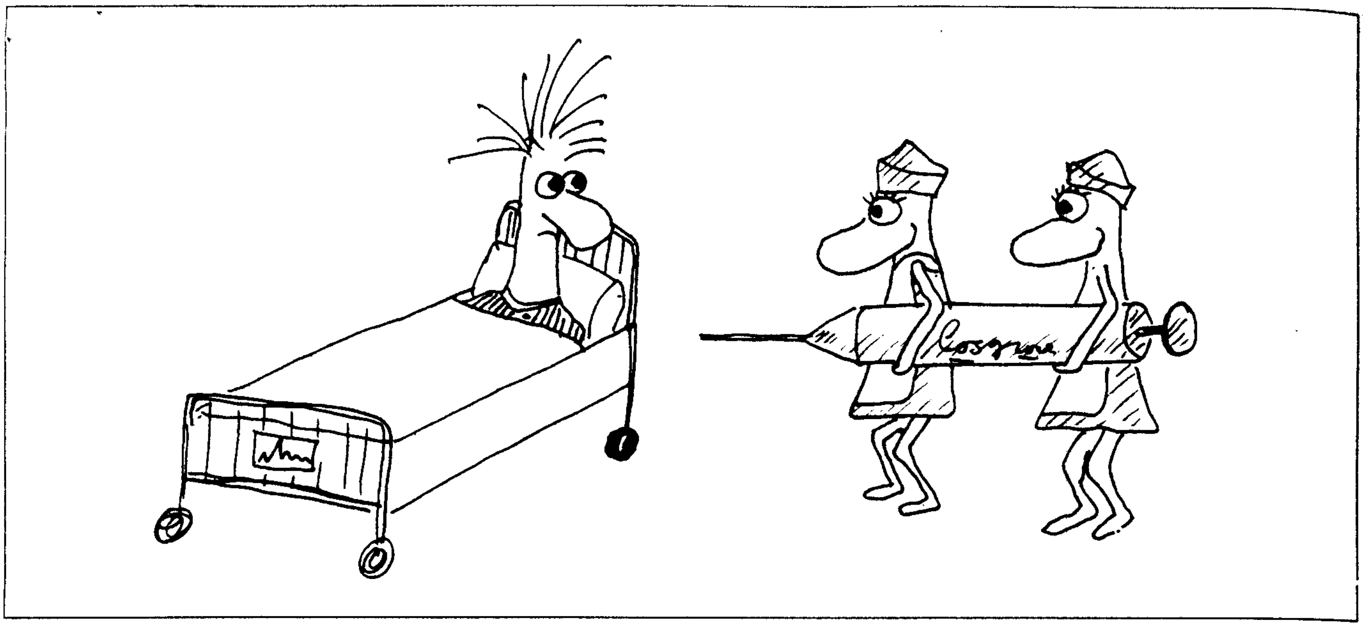
takes us back to the children $A_1, A_2 \ldots A_n$ (of Adelaide) who are offered—as penfriends—children $B_1, B_2 \ldots B_n$ (of Budapest). The ideal case, of course, is a bijective arrangement with ample choice of $n!$ selections if each child is given a different penfriend; but the left-hand side of our identity represents no such restriction. Suppose now that $A_1, A_2 \ldots A_k$ select different penfriends, but A_{k+1} chooses one who has already been selected by one of the first k, and that there are no restrictions on the choices made by $A_{k+2}, \ldots, A_n$. This gives $n\binom{n-1}{k-1}k!\ n^{n-k-1} = \binom{n-1}{k-1}n^{n-k}k!$ possibilities, since there are n choices for the penfriend of A_{k+1}, $\binom{n-1}{k-1}k!$ for the penfriends of $\{A_1, \ldots A_k\}$ and n^{n-k-1} for the remaining ones. The formula can be generalized to

$$m^n = \sum_{k=1}^{n-1}\binom{m-1}{k-1}m^{n-k}k! + m^{(n)},$$

the number of potential penfriends being m, not necessarily equal to n.

Can we treat sums of the squares, or cubes, or generally k-th powers of successive integers in a similar way? Consider a certain patient who is to be hospitalized for $n + 1$ days. If we go for the sum of the cubes of the first n natural numbers, then we make our patient undergo 4 medical tests, A, B, C and D, under the following conditions: A must precede all the other tests and take a full day. There is no restriction on B, C or D tests—they can be done in any order with any number of them on the same day. The hospital can schedule the tests by selecting the day for A first. If this is to take place on the k-th day, there are $(n + 1 - k)^3$ choices for the days of the other tests. Hence the number of choices is

$$\sum_{k=1}^{n}(n + 1 - k)^3 = \sum_{k=1}^{n} k^3.$$



Alternatively we must consider 3 cases:

(1) B, C and D are done on different days after A: $3!\binom{n+1}{4}$ choices;

(2) Two of B, C, D done the same day: $2.3\binom{n+1}{3}$ choices;

(3) B, C, D on the same day: $\binom{n+1}{2}$ choices;

altogether $3!\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} = n^2(n+1)^2/4$.

Similar reasoning, with only three tests, A, B and C, gives

$$\sum_{k=1}^{n} k^2 = 2\binom{n+1}{3} + \binom{n+1}{2} = \frac{(2n+1)(n+1)n}{6}.$$

Making our patient undergo $r+1$ tests in $n+1$ days yields an identity for $\sum_{k=1}^{n} k^r$, but the expression on the right-hand side involves Stirling numbers.

With a little variation we can obtain identities such as

$$1.\ (n-1)^2 + 2(n-2)^2 + 3(n-3)^2 + \ldots + (n-1).1^2 = \frac{n^2(n^2-1)}{12}$$

(The patient is subjected to tests A, B, C and D but this time both A and B take a full day, A is to precede B.) Varying the torture of our patient will produce further fanciful formulae of the convolutory type.

3. Alternating Sums

All the sums discussed up to this point contain positive terms only. There is, however, an abundance of combinatorial formulae involving sums, strictly alternating in sign. The main feature of all of these identities is the *Inclusion-Exclusion Principle* (I-E).

Let N be the number of objects endowed with some of the properties $\alpha_1, \alpha_2 \ldots \alpha_p$. Denote by N_{α_i} the number of objects having property α_i (($i = 1, \ldots, p$), not to the exclusion of some other properties) $N_{\alpha_i\alpha_j}$ the number of objects having both properties α_i, α_j (at least) ($i,j = 1, \ldots, p$) and so on, $N_{\alpha_1\alpha_2} \ldots \alpha_p$ being the number of objects having all the properties. *Then the number of objects having none of the listed properties is:*

$$N_0 = N - \sum_{i=1}^{p} N_{\alpha_i} + \sum_{\substack{i,j=1\\ i\neq j}}^{p} N_{\alpha_i\alpha_j} + \ldots + (-1)^p N_{\alpha_1\alpha_2} \ldots \alpha_p.$$

As a first example we turn again to the n Adelaide children with their m potential penfriends from Budapest. This time our condition is that no Budapest child should remain neglected. (Surjective mapping.) This is possible of course only if $m \leq n$. We use the I-E principle, writing $N = n^m$, representing all the possibilities and excluding the undesirable ones.

Let $N_{B_1}, N_{B_2} \ldots$ be the numbers of arrangements where B_1, B_2 etc. are neglected, $N_{B_1B_2}$ the number where B_1B_2 are both omitted and so on. Then

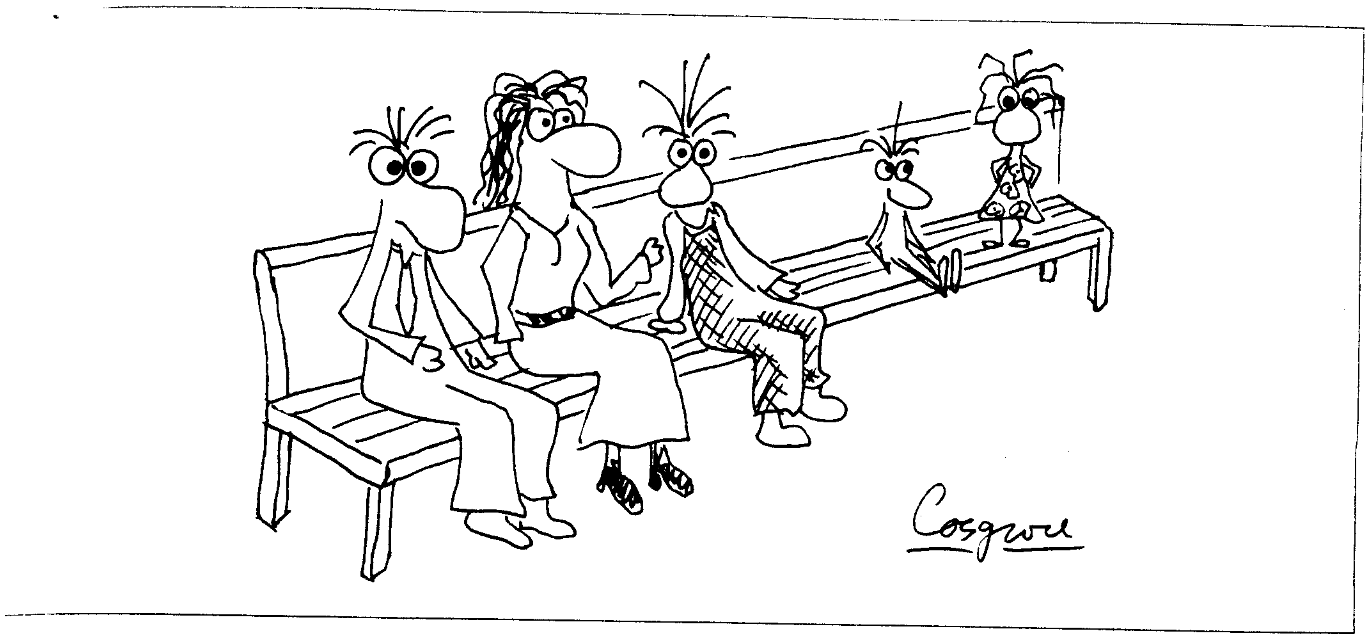
$$N_{B_1} = N_{B_2} = \ldots = N_{B_m} = (m-1)^n,$$
$$N_{B_1B_2} = N_{B_1B_3} = \ldots = (m-2)^n,$$
$$N_{B_1B_2} \ldots B_i = \ldots = (m-i)^n$$

the sum of terms of the last type being $\binom{m}{i}(m-i)^n$. Thus the number of desired arrangements is

$$\sum_{i=0}^{m-1} (-1)^i \binom{m}{i} (m-i)^n.$$

This formula immediately yields the identities

$$\sum_{i=0}^{m-1} (-1)^i \binom{m}{i} (m-i)^n = 0 \text{ when } n < m$$



$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^n = n!$$

$$\sum_{i=0}^{n-1} (-1)^i \, (m-i)^{m+1} = m\frac{(m+1)!}{2}.$$

One celebrated alternating sum formula gives the solution of the "problème des ménages". The number of possible seatings around a table of n couples so that no husband is to sit next to his wife is

$$N = 2n! \sum_{k=0}^{n} (-1)^k \, (n-k)! \, \frac{2n}{2n-k} \binom{2n-k}{k}.$$

The problem originates in the 19th century (Cayley, Lucas) with the formula given by Touchard and interpreted combinatorially by Kaplansky (1943).

The formula bears an interesting resemblance to the Chebyshev polynomial $T_n(x) = \cos n\,\theta$ where $x = \cos\theta$. In particular when $x = \cos(0) = 1$, the polynomial gives

$$2T_n(1) = 2 = \sum_{0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k}.$$

The common feature of the two formulae is the factor $n(n-k)\binom{n-k}{k}$ which has a combinatorial meaning. It gives the number of ways in which k objects can be placed in n slots *around a circle* so that the objects do not occupy adjacent slots. Allied to this is the expression $\binom{n-k+1}{k}$ which represents the number of ways in which k objects can be placed in n slots *along a line* so that two objects do not occupy adjacent slots. Both these formulae can be proved easily by using induction for the second one and then deducing the first one. They can also be interpreted by independent physical models (which are left to the reader as exercises).

We are going to use these two formulae together with the I-E principle to establish the Chebyshev expansion for $x = 1$ along with the allied identity

$$\sum_{0}^{\left[\frac{n}{2}\right]} (-1)^k \binom{n-k}{k} 2^{n-2k} = n+1.$$

Suppose that n seats are marked with A or C along a line or along a circle, respectively, to reserve seats for adults and children, with the restriction that *no child is to sit on the left side of an adult*. This of course includes seatings where all the seats are marked with A only, or C only. The obvious counts for these are then $n + 1$ along the line, and 2 around the circle (only A, or only C).

Now apply the I-E principle, denoting the number of unrestricted seating arrangements by N. We begin the numbering at the left of the line, or at any point of the circle, and denote by N_i the number of those arrangements where seat i is marked by C, but $i + 1$ is marked by A, thus violating the rule. N_{ij} is the number of those markings where the rule is broken both at i and at j, and so on. Clearly, *the seats at which the marking violates the rule cannot be adjacent.*

We see at once that $N = 2^n$, $N_i = 2^{n-2}$ for any i, since the marking of the seats other than i and $i + 1$

is left free (by the notation used in the I-E principle). Generally

$$N_{i_1 i_2 \ldots i_k} = 2^{n-2k}.$$

Next we find that

$$\sum N_{i_1 i_2 \ldots i_k} = \binom{n-k}{k} 2^{n-2k}$$

in the case of the line and

$$\sum N_{i_1 i_2 \ldots i_k} = \frac{n}{n-k}\binom{n-k}{k} 2^{n-2k}$$

in the case of the circle, using the respective formulae for the arrangement of k objects in k non adjacent slots, chosen out of n. The terms are now calculated; the general I-E formula gives both identities as stated.

Two questions arise naturally. Can this "method" be used for discovering new identities? Is it always applicable when we deal with finite combinations (sums, products, etc.) of integer valued expressions? The answer to the first question is certainly yes: counting different arrangements may yield new identities as we saw in some of the preceding examples. The second question must remain open and wait for some decidability formulation. While it is very rewarding to glimpse the combinatorial meaning on second sight, sometimes the hunt for it can become a strenuous and frustrating exercise (with no guarantee of success).

Here is a rather innocent looking example where the author has conceded defeat and invites the reader to try his bit. (No generating functions, please!)

$$4^n = \sum_{r=0}^{n} \binom{2r}{r}\binom{2n-2r}{n-r}.$$

Each side of this identity can be given a simple meaning, but where shall the twain meet?

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A Quote

"What's one and one and one and one and one and one and one and one and one and one?"

"I don't know," said Alice. "I lost count."

"She can't do addition," said the Red Queen.

Counting and Recounting: The Aftermath

Marta Sved

In my recent article (The Mathematical Intelligencer, 5.4 (1983), p. 21), I ended by challenging readers to provide a combinatorial proof to the identity:

$$4^n = \sum_{r=0}^{n} \binom{2r}{r} \binom{2n-2r}{n-r}.$$

This is a recount of the letters that I received from readers who continued where I left off by offering solutions to the problem.

The problem is not that new. P. Erdös, on reading the article, was quick to point out to me that Hungarian mathematicians tackled it in the thirties: P. Veress proposing and G. Hajos solving it. In his letter to me I. Gessel (M.I.T.) has given a survey of the more recent history of the problem. Proofs were published by D. Kleitman (Studies in Applied Mathematics 54. (1975), also by his student D. J. Kwiatowski (Ph.D. Thesis, MIT, 1975). It also found its way into texts (Feller, Mohanty).

In addition, I received solutions by A. Bondesen (Royal Danish School of Educational Studies, Copenhagen), K. Grünbaum (Roskilde Universitetscenter, Denmark), J. Hofbauer, jointly with N. Fulwick (Universität, Wien, Austria), D. Zeilberger (Drexel University, Philadelphia), and verbally from C. Pearce (Adelaide University), directly after reading the article.

All solutions are based, with some variations, on the count of lattice paths, or equivalently (1,0) sequences. Figure 1 is used to illustrate the simplest version. It represents a two-dimensional coordinate lattice, or a network of streets running East and North. We consider paths of length $2n$, beginning at O, proceeding in unit steps, heading East or North. It is clear that there are 2^{2n} ways in which a lattice point on the boundary AB can be reached. This gives the left-hand side of the identity.

Counting in a different way, assume that the last crossing of a path with the NE line (OM on the diagram) is at K(k,k), which of course may coincide with O or M. It is easy to see that there are $\binom{2k}{k}$ possible paths from O to K. Assuming for the moment that the number of ways the remaining $2n - 2k$ steps, (avoiding OM) may be taken, is similarly $\binom{2n-2k}{n-k}$, we obtain the desired right-hand side:

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k}.$$

The last statement, used to establish the identity, implies that the number of paths of length 2k, finishing at K (Fig. 2(a)), is the same as the number of paths of the same length, not touching the line OK (Fig. 2(b)). This is not obvious, but can be verified by manipulations involving binomial sums. An alternative geometric argument, coming from I. Gessel, is briefly sketched here.

As an intermediate step, consider paths of type shown in Fig. 2(c): these are completely in the upper half of the region, but may touch the line OK.

A path of type (a) may be transformed via (c) into type (b). The dotted line, drawn in Fig. 2(a), is a "tangent" to the path, parallel to OK and touching it for the first time at the extremity E. The transformation from (a) to (c) is done in three steps: cutting the path at E, translating the segment EK parallel to itself, bringing E to O and K to a point K', and finally fitting the OE segment, by placing the end-point E at K' and turning the segment to exchange vertical and horizontal directions, the image of the end originally at O coming to be the end of the path thus spliced together. A path of type (c) is thus obtained.

To get from (c) to (b) is necessary only if the path touches OK. In that case the horizontal unit-segment preceding the contact is turned vertical, the segment following it is shifted parallel to itself to the loose end, and finally this new path is reflected in OK into the lower region.

It can be shown easily that these "cutter, fitter, turner" operations from (a) to (c) and (c) to (b) have unique inverses (the reflection about OK in the second transformation ensures this). The composition of the two transformations gives a bijective map from (a) to (b).

A. Bondesen sent in an appealing variation of the theme of path-counting.

Fig. (3) represents "Polya-town" (Pascal *triangle in* disguise), the thick lines its streets, the circles, indexed by binomial coefficients, its corners. The thin lines with circles are the streets and corners of an under-

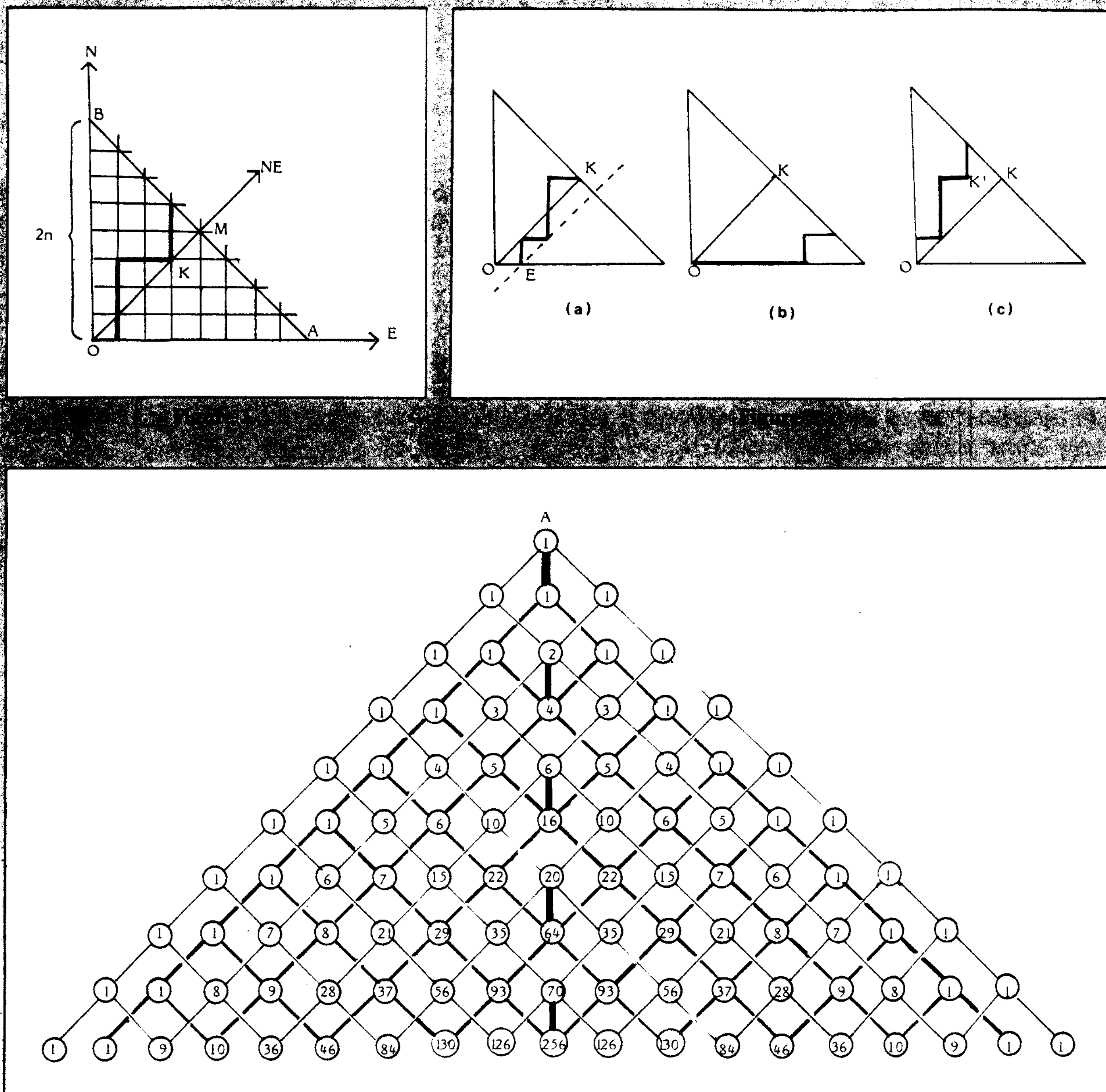

Figure 3

ground satellite town, while the heavy lines in the middle are ramps joining the two cities. The numbers indexing the corners of the satellite town are marked by the symbol $\hat{\binom{n}{k}}$, defined as sums of binomial coefficients belonging to points of Polya-town in the same row, as the satellite-town point considered and between it and the nearer river drive. It is shown that

$$\binom{n \,\hat{-}\, 1}{k-1} + \binom{n \,\hat{-}\, 1}{k} = \binom{\hat{n}}{k} + \delta^{n}_{2k}\binom{n}{k}$$

(δ the Kronecker symbol). The desired identity is interpreted as two counts of the shortest paths from A to the satellite point: $\hat{\binom{2n}{n}} = 2^{2n}$, using streets of either town together with the ramps.

We are now a (lattice)-step nearer to conjecturing that combinatorial identities can be proved combinatorially.

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