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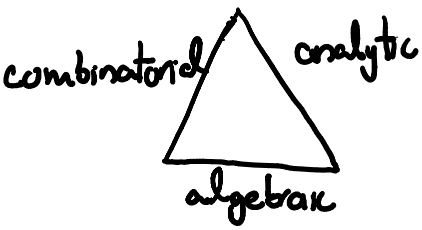
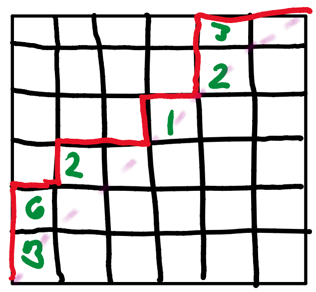
sábado, 22 de junho de 2019 09:58

Three faces of the delta conjecture



Combinatorial side:



$\square$ to diagonal ↓   see below for def-n ↓

a_i	d_i	b_i
1	0	0
0	0	1
0	1	0
1	2	1
1	2	3
0	0	0

d_i = # inversion pairs in row i and row above i

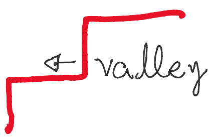
$b \diagup a \quad b<a$

Rise version

Now consider $\displaystyle\sum_{\pi}\sum_{\sigma} x^{\sigma} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\pi)} \prod_{\substack{a_i > a_{i-1} \\ i \geq 2}} \left(1 + \frac{z}{a_i}\right)$

π: Dyck path

σ: parking function



- Bottom of column. Moveable = pushing 1 to the left doesn't create $\boxed{\begin{smallmatrix} a \\ b \end{smallmatrix}}$ $a<b$

Valley version $\displaystyle\sum_{\pi}\sum_{\sigma} x^{\sigma} q^{\mathrm{dinv}(\sigma)} t^{\mathrm{area}(\pi)} \prod_{\substack{\text{moveable} \\ \text{valleys}}} \left(1 + \frac{z}{q^{d_i+1}}\right)$

↑
Symm. function?

Conj: Valley = Rise

Analytic side

t	tq	
1	q	q^2

μ a partition. $B_\mu(q,t) = \sum_{s \in \mu} q^a t^l$, e.g.

μ partition: $B_\mu(q,t) = \sum_{s\in\mu} q^{\cdot}t^{\cdot}$, e.g.

$$B_{3,2} = 1+q+q^2+t+tq$$

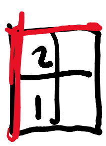
Now, for $f \in Sym$ $\quad \Delta_f \widetilde{H}_\mu = f[B_\mu]\widetilde{H}_\mu \qquad \widetilde{H}$ = modified Macdonald poly

$$\Delta'_f \widetilde{H}_\mu = f[B_\mu - 1]\widetilde{H}_\mu$$

e.g. $\Delta_{e_n} f = \nabla f$, $\Delta'_{e_{n-1}} f = \nabla f$

<u>Delta conjecture</u>: $\displaystyle\sum_{k=1}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n$ = Comb. version.

<u>Ex</u> $n=2$



$t(1+\frac{z}{t})x_1x_2$

$x_1^2 \qquad x_2^2 \qquad q x_1 x_2 \qquad x_1 x_2$

Get $z s_{(1^2)} + s_2 + (q+t)s_{(1^2)}$

<u>Compositional Shuffle Conjecture</u>

$$(C_m F)[X] = (-q)^{1-m} F\left[X + \frac{(1-q)}{qz}\right] \sum_{j=0}^{\infty} h_j[X] z^j \Big|_{z^m}$$

$\alpha \vDash n$ (composition) $\qquad C_\alpha := C_{\alpha_1} C_{\alpha_2} \cdots C_{\alpha_k}(1)$

$$\nabla C_\alpha = \sum_{\substack{\pi,\sigma \\ \text{touch}(\pi)=\alpha}} x^\sigma q^{\text{dinv}} t^{\text{area}}$$

Thm by Carlsson-Mellit

<u>Note</u> $\Delta'_{e_{k-1}} C_\alpha$ is not positive.

But D'Adderio, Iraci, Wyngaerd introduced the following

But D'Adderio, Iraci, Wyngaerd introduced the following compositional version of Δ-conj.

$\alpha \vDash n-k$

$$\Theta_k = \Pi e_k\left[\frac{X}{(1-q)(1-t)}\right]\Pi^{-1}, \quad \Pi \tilde{H}_\mu = \prod_{\substack{s\in\mu \\ s\neq(0,0)}}(1-q^{a'}t^{\ell'})\tilde{H}_\mu$$

Conj $$\Theta_k \nabla C_\alpha = \sum_{\substack{\pi \\ k \text{ decorated rises} \\ \text{touch}(\pi)=\alpha}} \sum_\sigma x^\sigma q^{\text{dinv}} t^{\text{area}}$$

Proof in progress. Uses $\Theta_k \nabla e_{n-k} = \Delta'_{e_{k-1}} e_n$

Example Let $F_n(q,t,w,z) = \sum_\pi q^{\text{dinv}} t^{\text{area}} \prod_{a_i > a_{i-1}}\left(1+\frac{z}{t^{a_i}}\right)\prod_{b_i > b_{i-1}}\left(1+\frac{w}{q^{b_i}}\right)$

b_i = # of dinv pairs involving i^{th} car in reading order and cars before it

Thm $F_n(q,t,w,z) = F_n(t,q,w,z) = F_n(q,t,z,w)$

At $z=0$, Schröder poly: Superpoly. Knot invariant in a,q,t that at $t=1/q$ specializes to HOMFLY.

For $w=-a$, $z=0$ $(1-a)$ times specializes to Schröder of $(n+1,n)$-torus knot

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Algebraic side

$$DR_n = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \Big/ \left\langle \sum x_i^a y_i^b \;\; \forall\, a+b>0 \right\rangle$$

$$DR_n = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n] \Big/ \langle \sum_i x_i^a y_i^b \ \forall\ a+b>0 \rangle$$

<u>Thm</u> (Haiman) $\nabla e_n = \mathrm{Frob}_{q,t}\, DR_n$.

$q=0$: HRS $\to$ family of quotient rings

<u>Zabrocki</u> $\qquad z_i z_j + z_j z_i = 0$

$$SDR_n = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_n] \Big/ \langle \sum x_i^a y_i^b z_i^c \ \ {\scriptstyle a+b+c>0,\ c\le 1} \rangle$$

"super diagonal coinvariants"

<u>Conj</u> $\displaystyle\sum_i z^i \theta_i \nabla e_{n-i} = \mathrm{Frob}\ SDR_n$

$$\sum_{i,j} z^i w^j \theta_i \theta_j \nabla e_{n-i-j} = \mathrm{Frob}\begin{pmatrix} \mathbb{C}[X,Y,Z,W]\ S_n\text{-coinv} \\ z_i z_j + z_j z_i = 0 \\ w_i w_j + w_j w_i = 0 \end{pmatrix}$$

Carlsson-Oblomkov: Basis of DR_n.

$\sigma = 2\ 5\ 7 \mid 1\ 3\ 8 \mid 4\ 6 \mid 0$ (max'l increasing subseq

$w = 3\ 3\ 2\ \ 2\ 1\ \ 2\ 2\ 1$

$w_i = \#\ \sigma_j > \sigma_i$ in σ_i's runner $+$

$\qquad \#\ \sigma_j < \sigma_i$ in next runner

[H, Loehr]

$$\sum_{\pi,P,F} q^{\mathrm{dinv}} t^{\mathrm{area}} = \sum_{\sigma\in S_n} t^{\mathrm{maj}\,\sigma} \prod_{i=1}^{n} [w_i(\sigma)]_q$$

Carlsson-Oblomkov basis:

union of all these $\begin{cases} y_2y_5y_7\, y_2y_5y_7\, y_1y_3y_8\,(1+x_2+x_2^2)(1+x_5+x_5^2)(1+x_7)(1+x_1) \\ \qquad (1+x_8)(1+x_4) \end{cases}$

union of all these $\left\{ y_2 y_5 y_7 \; y_2 y_5 y_7 \; y_1 y_3 y_8 \, (1+x_2+x_2^2)(1+x_5+x_5^2)(1+x_7)(1+x_1)(1+x_8)(1+x_4) \right.$

Setting $y_i=0$, get Artin basis for $\mathbb{C}[x]$-coinvariants

$x_i=0$, get Steinberg, Garsia-Stanton basis

<u>Thm</u> H, Sergel: $\sum_{PF} q^{dinv} t^{area} = \sum_{\substack{ord \\ sc.}} t^{minimaj} \prod [w_i(\sigma)]$

To express Procesi as Schur of tautological, coeff's are supposed to come from Δ-conjecture