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Quantizations of ~~Gieseker varieties~~ quantized Gieseker varieties & higher rank Catalan #s.

1) $r, n \in \mathbb{Z}_{>0}$ $\quad R = \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)$

$$\begin{array}{c} R \\ \circlearrowleft \\ G := GL_n(\mathbb{C}) \end{array}$$

$$T^*R = R \oplus R^* = \underset{(A,\, B,}{\mathrm{End}(\mathbb{C}^n)^{\oplus 2}} \oplus \underset{i}{\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)} \oplus \underset{,\ j)}{\mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)}$$

$G \curvearrowright T^*R$, w/ moment map

$$\mu : T^*R \longrightarrow \mathfrak{g}^* = \mathfrak{g}$$

$$\mu(A, B, i, j) = [A, B] - ji$$

$$\begin{tikzcd} \mu^* : \mathfrak{g} \arrow[r] \arrow[dr] & \mathbb{C}[T^*R] \\ & \mathrm{Vect}(R) \arrow[u] \end{tikzcd}$$

$$\xi \in \mathfrak{g} \longmapsto \xi_R \text{ - vector field}$$

- Hamiltonian reduction: $G \curvearrowright \mu^{-1}(0)$

$$\mathcal{M}(n, r) := \mu^{-1}(0) /\!\!/ G = \mathrm{Spec}\left[\left(\mathbb{C}[T^*R] \big/ \mathbb{C}[T^*R]\mu^*(\mathfrak{g})\right)^G\right]$$

Affine, singular, Poisson.

Resoln. of singularities: GIT quotient $\theta = \det : G \to \mathbb{C}^\times$

$$\mathcal{M}^\theta(n, r) = \mu^{-1}(0)^{\theta\text{-}ss} /\!\!/ G$$

nonzero

$\{$ no A & B stable subs? $($

$$= \left\{ (A,B,i,j) \in \mu^{-1}(0) \,\middle|\, \begin{array}{l} \text{no } A \,\&\, B \text{ stable subs.} \\ \text{in } \mathrm{Ker}(i) \end{array} \right\} \Big/ G$$

$\mathcal{M}^\theta(n,r)$ = smooth, irred. sympl. variety of dim $= 2nr$

$\mathcal{M}^\theta(n,r) \longrightarrow \mathcal{M}(n,r)$ - resoln of singularities

In what follows $\mathrm{End}(\mathbb{C}^n) \rightsquigarrow \mathfrak{sl}_n$

$\Rightarrow \dim = 2nr - 2$

Examples: $n = 1$

$\mathcal{M}^\theta(1,r) = T^*\mathbb{P}^{r-1}$, resolves $\mathcal{M}(1,r)$ = minimal nilp. orbit in $\mathfrak{gl}_r$

$r = 1$

$\mathcal{M}(n,1) = \mathfrak{h} \oplus \mathfrak{h}^*/S_n$, $\mathfrak{h} \cong \mathbb{C}^{n-1}$, refl. repn of S_n

$\mathcal{M}^\theta(n,1)$ = basically $\mathrm{Hilb}_n(\mathbb{C}^2)$

<u>Quantization</u> $\lambda \in \mathbb{C}$

$$\mathcal{A}_\lambda(n,r) = \left[\frac{\mathcal{D}(R)}{\mathcal{D}(R)\{\xi_R - \lambda \operatorname{tr}(\xi) \mid \xi \in \mathfrak{g}\}} \right]^G$$

assoc. algebra, w/ filtration by order of diff. op.

$$\operatorname{gr} \mathcal{A}_\lambda(n,r) = \mathbb{C}[\mathcal{M}(n,r)] = \mathbb{C}[\mathcal{M}^\theta(n,r)]$$

<u>Examples</u> $n=1$, $\mathcal{A}_\lambda(1,r) = D^\lambda(\mathbb{P}^{r-1})$

$r=1$, $\mathcal{A}_\lambda(n,1) =$ spherical RCA of type $\mathfrak{sl}_n$.

<u>Symmetries</u> $\mathbb{C}^\times \times GL(r) \curvearrowright R$

$$(z,h)(A,i) = (zA, hi)$$

commutes w/ G-action

$\Rightarrow \mathbb{C}^\times \times GL(r) \curvearrowright T^*R, \mathcal{M}(n,r), \mathcal{M}^\theta(n,r), D(R),$
$\mathcal{A}_\lambda(n,r)$

Hamiltonian actions

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2) Finite dim-l reps of $\mathcal{A}_\lambda(n,r)$

<u>Thm 1</u> 1) $\exists$ f.d. rep. of $\mathcal{A}_\lambda(n,r)$ iff $\begin{cases} \lambda = \frac{a}{n}, \ (a,n)=1 \\ \lambda \notin (-r,0) \end{cases}$

2) If so, $\mathcal{A}_\lambda(n,r)\text{-mod}_{fd} \simeq \text{Vect}$.

<u>Ex</u> $n=1$, $D^\lambda(\mathbb{P}^{r-1}) \twoheadleftarrow U(\mathfrak{sl}_r)$

$D^\lambda(\mathbb{P}^{r-1}) \curvearrowright \Gamma(\mathcal{O}(\lambda))$ $\quad \lambda \geq 0$

$r=1$, [Berest-Etingof-Ginzburg]

3) Construction of f.d. irrep

$$D(R)\text{-mod}^{G,\lambda} = \{ M \in D(R)\text{-mod with } G \curvearrowright M \mid \xi_M = \xi_R - \lambda \operatorname{tr}(\xi) \}$$

$$\mathcal{D}(R)\text{-mod}^{G,\lambda} = \{ M \in \mathcal{D}(R)\text{-mod with } G \circlearrowright M \mid \xi_M = \xi_R - \lambda\,\mathrm{tr}(\xi) \ \forall \xi \in \mathfrak{g} \}$$

$M \in \mathcal{D}(R)\text{-mod}^{G,\lambda}$

$$R(M) = \mathcal{M}^G \circlearrowleft \mathcal{D}(R)^G \text{ factors thru } \mathcal{A}_\lambda(n,r)$$

$\rightsquigarrow$ quotient functor $R : \mathcal{D}(R)\text{-mod}^{G,\lambda} \twoheadrightarrow \mathcal{A}_\lambda(n,r)\text{-mod}$

<u>Fact</u> $\exists!\ \mathcal{M}_{\frac{a}{n}} \in \mathcal{D}(\mathfrak{sl}_n)\text{-mod}^{SL_n}$ s.t.

$a>0,\ (a;n)=1$

- $\mathcal{M}_{\frac{a}{n}}$ is irred
- $\mathrm{Supp}(\mathcal{M}_{\frac{a}{n}}) \subseteq$ nilp. cone
- $\{(z,\dots,z) \mid z^n=1\}$ acts on $\mathcal{M}_{\frac{a}{n}}$ via the character $z \mapsto z^{-a}$

Intermediate ext from principal nilp. orbit!

$GL_n \circlearrowright \mathcal{M}_{\frac{a}{n}}$ w/ diag matrices acting via $z \mapsto z^{-a}$

$$\mathcal{M}_{\frac{a}{n}} \in \mathcal{D}(\mathfrak{sl}_n)\text{-mod}^{G,\lambda}$$

$$\rightsquigarrow \mathcal{M}_{\frac{a}{n}} \boxtimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)] \in \mathcal{D}(R)\text{-mod}^{G,\lambda}$$

<u>Propn</u> (Etingof-V.Krylov-L)

$$L_{a,n,r} := R\left(\mathcal{M}_{\frac{a}{n}} \boxtimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^r)]\right)$$

is irred. f.d. $\mathcal{A}_\lambda(n,r)$.

Goal: Compute $\dim L_{a,n,r}$.

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4) Higher rank Catalan #s.

Thm (Calaque-Enriquez-Etingof '07)

Mult. of G-irrep $V_n(\mu)$ ($\neq 0$ only if $\mu_1 + \dots + \mu_n = -a$) in $\mathcal{M}_{\frac{a}{n}}$

$$= \frac{1}{n} \dim V_n(\mu)$$

$$\mathbb{C}[\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)] = \bigoplus_{\nu} V_n(\nu) \otimes V_r(\nu)^*$$

ν = part w/ $\leq \min(n, r)$ rows

$$\Rightarrow L_{a,n,r} \underset{GL_r}{=} \bigoplus_{\substack{\nu, |\nu| = a \\ \leq \min(n,r) \text{ rows}}} V_r(\nu)^{\oplus \frac{1}{n} \dim V_n(\nu)}$$

Higher rank rational Catalan #

Corollary

$$\dim L_{a,n,r} = \frac{1}{n} \binom{nr + a - 1}{a}$$

For $r = 1$: $\frac{(n+a-1)!}{n!\, a!}$ ← rational Catalan #.

Adding stupid $\mathbb{C}^\times$ turns # into quantum #'s.

QUESTIONS

1) Combinatorial meaning? Basis in $L_{a,n,r}$ which is e-basis for certain

is e-basis for certain commutative sub. of $\mathcal{A}_\lambda(n,r)$

Coming from truncated shifted Yangian!

$\boxed{\text{Known for } r=1}$

2) Where's the clever 1-dim'l torus?

$$\mathcal{M}(n,r) \curvearrowleft GL_r \times \mathbb{C}^\times \times \mathbb{C}^\times$$

only gives a filtration

$\approx$ higher rank (q,t)-Catalan #s?

Way #1:- $\mathcal{M}_{\frac{a}{n}}$ has Hodge filtration $\rightsquigarrow$ gr...

Way #2:- Use EHA $\curvearrowright \bigoplus_{n=0}^{\infty} K_0\, Coh^{GL_r \times \mathbb{C}^\times \times \mathbb{C}^\times}(\mathcal{M}(n,r))$

apply generator of slope $\frac{a}{n}$ to $|\emptyset\rangle$.