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On G-equivariant quantizations of nilpotent coadjoint orbits

1) Quantizations of Poisson algebras

A = f.g Poisson algebra, graded $A = \bigoplus_{i \geq 0} A_i$, $A_0 = \mathbb{C}$
$\{\cdot,\cdot\}$ of deg $= -d$, $d \in \mathbb{Z}_{>0}$

Quantization: $\mathcal{A}_\hbar$ = $\mathbb{C}[[\hbar]]$-algebra, flat/$\mathbb{C}[[\hbar]]$,
complete & separated in $\hbar$-adic topology

$\mathbb{C}^\times \curvearrowright \mathcal{A}_\hbar$ - rational on $\mathcal{A}_\hbar/(\hbar^k)$ $\forall k$ $t.\hbar = \hbar$

$[\mathcal{A}_\hbar, \mathcal{A}_\hbar] \subseteq \hbar^d \mathcal{A}_\hbar \Rightarrow$ new bracket $\{a,b\} = \frac{1}{\hbar^d}[a,b]$

$\theta: \mathcal{A}_\hbar/(\hbar) \xrightarrow{\sim} A$ of Poisson graded algebras

So, quantization is the pair $(\mathcal{A}_\hbar, \theta)$.

<u>Example</u> $\mathfrak{g} = s/s$ Lie algebra, $A = S(\mathfrak{g})$, $\deg \mathfrak{g} = 2$,

$$\mathcal{U}_\hbar(\mathfrak{g}) = T(\mathfrak{g})[[\hbar]] / (\xi \otimes \eta - \eta \otimes \xi = \hbar^2[\xi,\eta])$$

Taking locally finite vectors + modding out $\hbar - 1$, we get usual enveloping algebra

2) Quant. of Poisson schemes

$X/\mathbb{C}$ is a Poisson scheme if $\mathcal{O}_X$ is endowed w/ Poisson bracket $\{\cdot,\cdot\}$.

Ex $\mathfrak{g}^*$, Spec(Poisson algebra), symplectic varieties

Quantization $(\mathcal{D}, \theta)$, $\mathcal{D}$ is a sheaf on X of $\mathbb{C}[[\hbar]]$-alg. w/ isomorphism

$$\theta : \mathcal{D}/\hbar\mathcal{D} \xrightarrow{\sim} \mathcal{O}_X$$

Example of interest: $X = \mathbb{O} \subseteq \mathfrak{g}^*$ - nilp. orbit w/ Kostant-Kirillov sympl. form

We say that $(\mathcal{D}, \theta)$ is G-equivariant if $G \curvearrowright \mathcal{D}$, $\hbar$ is G-inv, θ is G-equivariant

Thm (M) If $\mathcal{D}$ is a quant of X ($X = \mathbb{O}$), then $\Gamma(\mathcal{D})$ is a quantization of $\mathbb{C}[X] := \Gamma(X, \mathcal{O}_X)$.

for $\mathfrak{g}$ being simple & classical

Note that we get an embedding from

$$0 \to \hbar\mathcal{D} \to \mathcal{D} \to \mathcal{O}_X \to 0$$

$$\Rightarrow \Gamma(\mathcal{D})/(\hbar) \hookrightarrow \mathbb{C}[X].$$

More geometrically, X embeds into its affinization

$$i : X \hookrightarrow \operatorname{Spec} \mathbb{C}[X].$$

and the claim is that we have inverse bijections

$$\operatorname{Quant}^G(X) \underset{i^{-1}}{\overset{i_*}{\longleftrightarrow}} \operatorname{Quant}(\operatorname{Spec} \mathbb{C}[X]).$$

$$\operatorname{Spec} \mathbb{C}[X] \overset{\text{Losev}}{=} \text{Affine space}$$

3) Towards the proof

$G \curvearrowright \mathcal{D}$ w/ quantum comoment map

$\Phi : \mathcal{U}_\hbar(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D})$, so killing the kernel

$$\mathcal{U}_\hbar(\mathfrak{g})/J \hookrightarrow \Gamma(\mathcal{D}).$$

Now let $B = (\mathcal{U}_\hbar(\mathfrak{g})/J)/(\hbar)$, $Y = \operatorname{Spec}(B) \subseteq \mathfrak{g}^*$

Note that $Y_{red} = \overline{X}$

<u>Prop</u> X is a normal, CM Poisson scheme w/ finitely many symp leaves and $X_2 = X^{reg} \cup$ all codim 2 leaves. Then any quant of X_2 extends uniquely to a quant of X.

<u>Note</u> In our case, $\operatorname{Spec} \mathbb{C}[X]$ satisfies these conditions

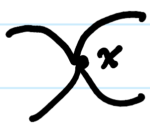
Note: In our case, Spec $\mathbb{C}[X]$ satisfies these conditions

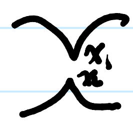
$$\begin{array}{c} \mathrm{Spec}(\mathbb{C}[X]) \\ \downarrow \\ \overline{X} \subseteq \text{Nilpotent cone} \end{array}$$

eg classical

<u>Thm</u> (Kraft-Procesi) Sing of $\mathcal{O}' \underset{\text{codim } 2}{\subseteq} \overline{\mathcal{O}}$ is either:

- Kleinian sing of type A, A_k } Lifting quantization is
- Kleinian sing of type D, D_k } easy here
- $A_k \cup_{pt} A_k$ ← need to work for this one.

$X =$ 

$\tilde{X} =$ 

$\mathcal{D}$ quant   Wts. $\tilde{\mathcal{D}}$ quant

On stalks

$$\begin{tikzcd}
\mathcal{D}_x \arrow[r] \arrow[d] & \tilde{\mathcal{D}}_{x_1} \oplus \tilde{\mathcal{D}}_{x_2} \arrow[d, "\text{← need to show surj-ve}"] \arrow[dr, bend left, "\text{enough to show surj-ve which is doable}"] & \\
\mathcal{O}_{X,x} \arrow[r] & \mathcal{O}_{\tilde{X},x_1} \oplus \mathcal{O}_{\tilde{X},x_2} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

And this essentially shows the thm