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(jt. wt. Brendan Pawlowski & Andy Wilson)

Plan: (I) Flag variety
(II) Spanning configurations.
(III) Δ, δ

(I) $\mathbb{Q}[x_1, \ldots, x_n] / (\mathbb{Q}[x_1, \ldots, x_n]^{S_n}_+) =: R_n$ graded S_n-module $\mathbb{Q}$-algebra

<u>Facts</u> ① [Chevalley] $R_n \cong_{S_n} \mathbb{Q}S_n$

② [E. Artin] $\mathrm{Hilb}(R_n; q) = [n]!_q$

③ [Lusztig-Stanley] $\mathrm{grFrob}(R_n; q) = \sum_{T \in SYT(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}$

$\mathcal{F}\ell_n = GL_n(\mathbb{C})/B = \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid \dim V_i = i\}$

<u>Facts</u> ① $\mathcal{F}\ell_n = \coprod_{w \in S_n} BwB/S_n$ - a CW dec of $\mathcal{F}\ell_n$ [Ehresmann]

② $H^\bullet \mathcal{F}\ell_n = R_n^{\mathbb{Z}}$, $x_i \leftrightarrow -c_1(\mathcal{V}_i/\mathcal{V}_{i-1})$

For $k \leq n$, $S_n \curvearrowright OP_{n,k} = \left\{ \begin{array}{l} \text{all ordered set partitions} \\ \text{of } \{1, \ldots, n\} \text{ into } k \text{ blocks} \end{array} \right\}$

e.g. $(4 \mid 156 \mid 23) \in OP_{6,3}$

<u>Def</u> [Haglund-R-Shimozono] For $k \leq n$:

$R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n] / \langle e_n, e_{n-1}, \ldots, e_{n-k+1}, \rangle$

$$R_{n,k} := \mathbb{Q}[x_1,\dots,x_n] \Big/ \left\langle \begin{matrix} e_n, e_{n-1}, \dots, e_{n-k+1}, \\ x_1^k, \dots, x_n^k \end{matrix} \right\rangle$$

graded $\mathbb{Q}$-algebra $\qquad R_{n,n} = R_n$

graded S_n-module

<u>Factn</u> [HRS] ① $R_{n,k} \cong_{S_n} \mathbb{Q}[OP_{n,k}]$ reverse coefficients

② $\mathrm{Hilb}(R_{n,k};q) = \mathrm{rev}_q\left([k]_q \, \mathrm{Stir}_q(n,k)\right)$

q-Stirling #'s: $\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1,k-1) + [k]_q \mathrm{Stir}_q(n-1,k)$

eg $n=3, k=2$: $1+3q+2q^2$: <u>NOT</u> palindromic

③ $\mathrm{grFrob}(R_{n,k};q) = \sum\limits_{T\in SYT(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n-\mathrm{des}(T)-1 \\ n-k \end{bmatrix}_q S_{\mathrm{shape}(T)}$

<u>Goal</u> Want $X_{n,k}$ s.t. $H^*(X_{n,k}) = R^{\mathbb{Z}}_{n,k}$ (not palindromic!)

<u>Def</u> [Pawlowski-R] For $k \le n$, (Spanning line conf)

$$X_{n,k} = \left\{ (\ell_1,\dots,\ell_n) \,\middle|\, \ell_i \subseteq \mathbb{C}^k \text{ is a line}, \ \ell_1 + \dots + \ell_n = \mathbb{C}^k \right\}$$

$X_{n,1} = \{*\}$, $X_{n,n} = G/T \overset{\text{htpy eq}}{\twoheadrightarrow} G/B = \mathcal{F}\ell_n$

$X_{n,k} \subseteq (\mathbb{P}^{k-1})^n$, Zariski open

<u>Facts</u> [PR] ① $X_{n,k}$ admits affine paving indexed by $OP_{n,k}$

② $H^{\bullet}(X_{n,k}) = R^{\mathbb{Z}}_{n,k}$, $\quad x_i \longleftrightarrow c_1(\ell_i^*)$

Need clever affine paving on $(\mathbb{P}^{k-1})^n$
[PR]: Schubert poly in this context.

$N = \alpha_1 + \cdots + \alpha_n$

<u>Rmk</u> [R]. Fix $k > 0$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{1, \ldots, k\}^n$

$$X_{\alpha,k} := \{(W_1, \ldots, W_n) \mid W_i \in Gr(\alpha_i, k),\ W_1 + \cdots + W_n = \mathbb{C}^k\}$$

$$H^{\bullet}(X_{\alpha,k}) = \left(\mathbb{Z}[x_1, \ldots, x_N]/I\right)^{S_\alpha},$$

$I =$ gen'd by $e_N, \ldots, e_{N-k+1}$,

$h_k, h_{k-1}, \ldots, h_{k-\alpha_i}$ in i^{th} batch of α_i variables

$x_1, \ldots, x_N \leftrightarrow$ Chern roots of $W_1^* \oplus \cdots \oplus W_n^*$

No 0 rows!

e.g. $\alpha = (2, 1, 2, 1)$ $\quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \updownarrow k$

<u>Delta conjecture</u> [Haglund-Remmel-Wilson] Fix $k \le n$,

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(X; q, t) = \mathrm{Valley}_{n,k}(X; q, t)$$

<u>Known</u> [Garsia, Haglund, R, Yoo, Remmel, Shimozono...]

$$\Delta'_{e_{k-1}} e_n |_{t=0} = \mathrm{Rise}_{n,k}(X; q, 0) = \mathrm{Rise}(X; 0, q) =$$

$$\Delta'_{e_{k-1}} e_n \big|_{t=0} = \underset{\text{Valley}}{\mathrm{Rise}_{n,k}}(X;q,0) = \mathrm{Rise}(X;0,q) = \underset{=\ \text{Valley}}{}$$

Fact [HRS] $\mathrm{grFrob}(R_{n,k};q) = (\mathrm{rev}_q \circ \omega)\Delta'_{e_{k-1}} e_n \big|_{t=0}$

$$= \mathrm{grFrob}(H^{\cdot}(X_{n,k};\mathbb{Q});\sqrt{q})$$

$\mathbb{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n,\theta_1,\ldots,\theta_n] = \mathbb{Q}[X_n,Y_n,\Theta_n]$

$\circlearrowleft$ S_n

Conj (Zabrocki) $\mathrm{grFrob}\Big(\overbrace{\mathbb{Q}[X_n,Y_n,\Theta_n] \big/ \mathbb{Q}[X_n,Y_n,\Theta_n]^{S_n}_{+}}^{DSR_n} ; \underset{x\ \ y\ \ \theta}{q,t,z}\Big)$

$$\| $$

$$\sum_{k=1}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n$$

Salomon

$\mathbb{Q}[X_n,\Theta_n]$, $SR_n = \mathbb{Q}[X_n,\Theta_n] \big/ \langle \mathbb{Q}[X_n,\Theta_n]^{S_n}_{+} \rangle$

"Ω_n"

Still open: $\mathrm{grFr}(SR_n;q,z)$

$$\overset{?}{=} \sum_{k=1}^{n} z^{n-k} \Delta'_{e_{k-1}} e_n \big|_{t=0}$$

Def [R-Wilson] $k \le n$, $\delta_{n,k} \in \mathbb{Q}[X_n,\Theta_n]$

"Super Vandermonde": Let $r = n-k$

$$\delta_{n,k} = \varepsilon_n\left(x_1^{k-1} x_2^{k-1} \cdots x_r^{k-1} x_{r+1}^{k-1} x_{r+2}^{k-2} \cdots x_{n-1}^{1} x_n^{0} \theta_1 \cdots \theta_r\right)$$

$$\delta_{n,k} = \varepsilon_n (x_1 \, x_2 \cdots x_r \, x_{r+1} \, x_{r+2} \cdots x_{n-1}^{1} \, x_n \, \theta_1 \cdots \theta_r)$$

$$\varepsilon_n = \sum_{S_n} (\mathrm{sgn}\, w) w$$

$$V_{n,k} = \mathbb{Q}\text{-span} \left\{ \left(\frac{\partial}{\partial x_1}\right)^{b_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{b_n} \delta_{n,k} : b_i \geq 0 \right\}$$

graded S_n-module

FACT [RW] $\mathrm{grFrob}(V_{n,k}; q) = \Delta'_{e_{k-1}} e_n \big|_{t=0}$

Polarization ops. on $\mathbb{Q}[X_n, Y_n, \Theta_n]$, $j = 1, 2, 3 \ldots$

$$P^{(j)}_{x \to y} = y_1 \left(\frac{\partial}{\partial x_1}\right)^j + \cdots + y_n \left(\frac{\partial}{\partial x_n}\right)^j$$

Def [RW] $\mathbb{V}_{n,k}$ = smallest linear subspace of $\mathbb{Q}[X_n, Y_n, \Theta_n]$, containing $\delta_{n,k}$ and closed under

- $\partial/\partial x_i$, $\partial/\partial y_i$
- $P^{(j)}_{x \to y}$, $j = 1, 2, \ldots$

Conj $\mathrm{grFrob}(\mathbb{V}_{n,k}; q, t) = \Delta'_{e_{k-1}} e_n$