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1: Affine Hecke Algebras G: conn'd red've grp

(V1) $\mathcal{H}_{aff} = \{f : G(\mathbb{F}_q((t))) \to \mathbb{C} \mid \text{f is compactly supp, bi-invariant w.r.t } I\}$

$$I = \{x \in G(\mathbb{F}_q[[t]]) \mid x(0) \in \text{Borel}\}$$

$K = \mathbb{F}_q((t))$, $\mathcal{O} = \mathbb{F}_q[[t]]$

$I \backslash G(K) / I \leftrightarrow W_{aff} = W \ltimes \mathbb{Z}^n$ extended affine Weyl grp.

Mult: Convolution of functions

$$(f \star f')(x) = \int_{G(K)} f(xy^{-1}) f'(y)\, dy$$

(V2) Generators & Relations

(W_{aff}, S) is an almost Coxeter grp

Thm (Iwahori) A v-deformation of presentation of (W_{aff}, S) defines a $\mathbb{Z}[v^{\pm 1/2}]$-alg

$$\mathcal{H}(W_{aff}, S)|_{v=q} = \mathcal{H}_{aff}$$

__Related__ Spherical Hecke alg-s.

$$\mathcal{H}_{sph} = \{ f : G(K) \to \mathbb{C} \,/\, \text{bi-inv. w.r.t } G(\mathcal{O}) \}$$

__Facts__ ① Let $G^\vee$ = Langlands dual of G

(Satake) $\mathcal{H}_{sph} \overset{\text{ring}}{\cong} K_0(\mathrm{Rep}\, G^\vee)$

(Bernstein) $\mathcal{H}_{sph} = Z(\mathcal{H}_{aff})$

Categorify!

#1.- [Mirković - Vilonen]

#2.- ??

Idea: functions $\longleftrightarrow$ Sheaves

equivariance Variety

$G(\mathcal{O}) \backslash G(K) / G(\mathcal{O})$
or I or I

$\mathcal{F}\ell = G(K)/I$ affine flag vty
$Gr = G(K)/G(\mathcal{O})$ affine Grassmannian

mult. $\rightsquigarrow$ convolution $\rightsquigarrow$ derived setting

(VI) Deligne's Mixed ℓ-adic sheaves } a LOT of work that has been done already

$D^b_{I,\mathrm{mixed}}(\mathcal{F}\ell, \overline{\mathbb{Q}}_\ell)$ Tate twist

$D_{I,\text{mixed}}(\mathcal{F}l, \mathbb{Q}_\ell)$ "Tate twist already

$\Downarrow$

$D^b_{G(O),\text{mixed}}(Gr, \overline{\mathbb{Q}_\ell})$ action of γ.

(V2) Take $\mathbb{C}$-version of $\mathcal{F}l$, Gr. k - any field

$D^b_I(\mathcal{F}l, k)$ convolution $\to$ $\mathbb{Z}[W_{aff}]$ ☹

So we need smth else

$\text{Parity}_I(\mathcal{F}l, k) \subseteq D^b_I(\mathcal{F}l, k)$

Parity sheaves: Even $\oplus$ Odd

stalks/costalks are concentrated in even degrees

Observations/Examples

① $\underline{k}_{pt}$ "skyscraper" $\mathbb{1}$.

$\underline{k}_{pt}[n]$ is parity

For a reflection s, the Iwahori orbit is $\mathbb{P}^1$

$\underline{k}_{\mathbb{P}^1}[n]$ - parity

$\underline{k}_{\mathbb{P}^1}[1]$ - perverse (an IC sheaf)

② [Juteau-Mautner-Williamson] $\mathrm{Parity}_I(\mathcal{F}l, k)$ is preserved by convolution

③ <u>Prop</u> $\mathrm{Parity}_I(\mathcal{F}l, k)$ categ $\mathcal{H}_{aff}$
$\mathrm{Parity}_{G(O)}(Gr, k)$ categ $\mathcal{H}_{sph}$

④ If $k = \mathbb{C}$, these are called semisimple complex.

⑤ Char $k = 0$, get KL basis
Char $k = \ell$, get ℓ-canonical basis

<u>Thm</u> [Gaitsgory] Nearby cycles allows a central functor

$$\psi : \mathrm{Perv}_{G(O)}(Gr, k) \to D^b_I(\mathcal{F}l, k)$$

(a) So this doesn't quite catfy Bernstein's thm/$\mathbb{C}$
(b) Working/$\mathbb{F}_q$ with ℓ-adic, mixed sheaves we're OK

<u>Note</u> For (a), $\psi(\text{parity}) \neq \text{parity}$
This is the problem we need to overcome

<u>Defn</u> [Achar-Riche]

$$D^{mix}_I(\mathcal{F}l, k) := K^b(\mathrm{Parity}_I(\mathcal{F}l, k))$$

"mixed modular derived cat-y". Properties:

"mixed modular derived cat-y". Properties:

① perverse t-structure (recollement/gluing)
② notion of Tate twist.
③ ~~Grothendieck six functors~~
④ [Achar] Nearby cycles

#3 - Detour: Nearby cycles

$$\begin{array}{c} X \\ \downarrow f,\ \mathrm{alg} \\ \mathbb{C} \end{array} \qquad X_0 = f^{-1}(0), \quad X_\eta = f^{-1}(\mathbb{C}^\times)$$

E.g.

$$\begin{array}{c} X = \mathbb{C}^n \\ \downarrow f = \text{mult. coord.} \\ \mathbb{C} \end{array} \qquad X_\eta = (\mathbb{C}^\times)^n, \quad X_0 = \sqcup \text{ coord hyperplanes}$$

Nearby cycles: $\Psi_f : D^b(X_\eta) \to D^b(X_0)$

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook'] \arrow[d] & \widetilde{X}_\eta \arrow[l, "\exp_X"'] \arrow[d] \arrow[lll, bend right] \\
0 \arrow[r] & \mathbb{C} & \mathbb{C}^\times \arrow[l] & \mathbb{C} \arrow[l, "\exp"']
\end{tikzcd}$$

$$\Psi_f := i^* j_* \exp_{X*} \exp_X^*$$

monodromy action $\Rightarrow \Psi_f = \Psi_f^{\text{unipotent}} \oplus \Psi_f^{\text{semisimple}}$

Thm [Achar] Assume Parity(X) makes sense, $\mathbb{C}^\times$ action compatibly on $X \xrightarrow{f} \mathbb{C}$, + more

Then, $\exists$ unipotent nearby cycles

$$\Psi_f^{un} : D^{mix}_{\mathbb{G}_m}(X_\eta, k) \longrightarrow D^{mix}(X_0, k)$$

$$\begin{tikzcd} \mathcal{F}\ell \arrow[r, hook] \arrow[d] & X \arrow[d] & Gr \times \mathbb{C}^\times \arrow[l, hook'] \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{C} & \mathbb{C}^\times \arrow[l, hook'] \end{tikzcd}$$

X = global affine Grassmannian (Zhu)

$\mathcal{G}$ = Iwahori Group Scheme, interpolate btw I and $G(\mathcal{O})$

$X(R) = \{(y, \mathcal{E}, \beta)\}$

y: R-point of $\mathbb{A}^1$; $\mathcal{E}$: $\mathcal{G}$-bundle; β: trivialization

Eg 3 $G = PGL_n$,

X = "global Schubert vty. assoc. to 1st fund coweight ϖ_1"
$= \overline{Gr}_{\varpi_1}$.

$$\begin{tikzcd} X \arrow[d, "f"] \\ \mathbb{C} \end{tikzcd}$$

$t \neq 0 \Rightarrow f^{-1}(t) = \mathbb{P}^1$

$f^{-1}(0) \subseteq \mathcal{F}\ell$ "central deg"

Thm [Achar-R] Explicitly compute $\Psi_f^{un}(k_{X_\eta}[n])$ for eg 1 & 3, as complex of parity sheaves.

Thm (Achar–?) Explicitly compute $\pi_*(\mathbb{Z}_{X_n}[?])$ to eg 1 & J, as complex of parity sheaves

coord hyp

Key: $\exists$ open $\mathbb{C}^? \subseteq \overline{Gr_v}$ compatible w/ $\mathbb{P}^1_n$

Conjecturally coincide w/ Beil's