HC bimodules for quantized Hilbert schemes

1) Quantized Hilbert schemes

Type A rational Cherednik algebras (type $\mathfrak{gl}_n$)

$c \in \mathbb{C}$

[Etingof-Ginzburg ea 2000] $H_c := \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \rtimes S_n \Big/ \begin{matrix} [x_i, x_j] = 0, \ [y_i, y_j] = 0 \\ [y_i, x_j] = c(ij) \\ [y_i, x_i] = 1 - c\sum_{j \neq i} (ij) \end{matrix}$

appeared in Minh-Tran Tinh

eg $c = 0$, $H_0 = \mathcal{D}(\mathbb{C}^n) \rtimes S_n$

$\mathbb{C}S_n \subseteq H_c$, $e = \frac{1}{n!} \sum_{w \in S_n} w$ idemp in H_c

eH_ce is almost the algebra $\mathcal{A}(n,1)$ that appeared in I. Losev's talk

- filtered, $F_0 H_c = \mathbb{C}W$, $F_1 H_c = \mathbb{C}W, \mathbb{C}\{x_i w, y_i w \mid {}^{w \in W}_{i = 1, \dots, n}\}$

$$F_k H_c = (F_1 H_c)^k$$

$$\mathrm{gr}^F H_c = \underset{\cup\!\mid}{\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]} \rtimes S_n$$

center $\quad Z = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}$

- graded, $\deg x_i = 1$, $\deg y_i = -1$, $\deg w = 0$

inner: $\quad eu = \frac{1}{2} \sum x_i y_i + y_i x_i$

Rep. theory of H_c has been focused on a cat-y of highest weight modules

$\mathcal{O}_c$ [Ginzburg-Guay-Opdam-Rouquier]

- highest weight cat-y

$P_c(\lambda) \twoheadrightarrow \Delta_c(\lambda) \twoheadrightarrow L_c(\lambda)$ $\lambda \in \mathrm{Irrep}\, S_n = \mathrm{Partition}($

proj-ve standard simple

Rmk eu acts semisimply on $\Delta_c(\lambda), L_c(\lambda)$

[Rouquier, Losev] $\mathcal{O}_c \simeq S_q(n)$-mod + q-Schur algebra $q = \exp(2\pi\sqrt{-1}c)$

2) HC bimodules

$c, c' \in \mathbb{C}$ $\quad gr H_c = gr H_{c'}$

[Berest-Etingof-Ginzburg '03, Losev '12] **Def** An $(H_c, H_{c'})$-bimodule $\mathcal{B}$ is called HC if it admits a filtration s.t.

- $gr \mathcal{B}$ is a f.g. $\mathbb{C}[\mathfrak{h}, \mathfrak{h}^*]^{S_n}$-bimodule
- the left and right actions of Z on $gr \mathcal{B}$ coincide

$SS(\mathcal{B}) := supp(gr \mathcal{B}) \subseteq Sym^n \mathbb{C}^2$ $\quad$ [totally analogous to HC $U(\mathfrak{g})$-bimodules]

Examples 1) H_c is a HC H_c-bimodule

2) Subquotients (easy) and extensions (hard) of HC bimodules are HC.

3) Translation bimodules are HC

$HC(c, c')$ - catg of HC bimodules ($\otimes$ catg if $c = c'$)

Want Describe this catg $\quad$ [**Note** equivalent to catg of modules / f.d. algebra]

Rmk If $\mathcal{B} \in HC(c, c')$, $M \in \mathcal{O}_{c'} \Rightarrow \mathcal{B} \otimes_{H_{c'}} M \in \mathcal{O}_c$

Thm (S.) $HC(c, c') = 0$ unless

1) $c - c' \in \mathbb{Z}$ or $c + c' \in \mathbb{Z}$, or

2) $c = \frac{r}{m}$, $c' = \frac{r'}{m}$ $\gcd(r, m) = \gcd(r', m) = 1$ and $m | n$.

In case ①, $D^b_{HC}((H_c, H_{c'})\text{ bimod}) \simeq D^b_{HC}(H_c\text{-bimod})$

In case ②, but not ①, $HC(c, c') \cong S_{\frac{n}{m}}\text{-rep}$

③ HC vs. GGOR

use $\Delta(\mathrm{sign})$ for $c<0$.

$$\Phi_c : HC(c,c) \longrightarrow \mathcal{O}_c$$
$$B \longmapsto B \otimes_{H_c} \Delta(\mathrm{triv})$$

Type A only!

Thm (S.) Φ_c is a exact fully faithful embedding whose image is closed under subquotients. [Analogous to ~~???~~ Bernstein-Gelfand thm for enveloping algebras]

- Simples in the image of Φ_c: $\boxed{\text{Mention ~~???~~ ~~???~~ we're left!}}$

very boring

$c \notin \{\frac{r}{m} \mid 1 \le m \le n, \cancel{\gcd(r,m)=1}\} \Rightarrow$ the only simple in the image of Φ_c is $L(\mathrm{triv}) = \Delta(\mathrm{triv})$
$\Rightarrow HC(c,c) \overset{\otimes}{\cong} \mathrm{Vect}$

$c \in \mathbb{Z} \Rightarrow \Phi_c$ is an equivalence [BEG], $HC(c,c) \overset{\otimes}{\cong} S_n\text{-rep}$

(…ser partial …z function Wilcox, Losev-Shelley -Afordhanov)

$c = \frac{r}{m}$, $1 < m \le n$, $\gcd(r,m)=1$. $L_c(\lambda) \in \mathrm{Im}\,\Phi_c$ iff in the dec $\lambda = \mu + m\nu$, μ is m-trivial

$\lambda \vdash n \Rightarrow \exists!$ decomposition $\lambda = \mu + m\nu$, μ is m-regular ($\mu_i - \mu_{i-1} < m$)

A partition μ is called m-trivial iff $\mu = ((m-1)^a, b)$, $a \ge 0$, $0 \le b < m-1$

$H_c = \mathrm{Hom}_{\mathcal{O}_c}(\Delta_c(\mathrm{triv}), \Delta_c(\mathrm{triv}))$

$D_c = \mathrm{Hom}_{\mathcal{O}_c}(\Delta_c(\mathrm{triv}), \nabla_c(\mathrm{triv}))$

double wall-crossing bimodule [Bezrukavnikov-Losev]

Examples ① $n=m$ $\quad L_c((n-1,1))$ $(\mu=(n-1,1),\ \nu=\emptyset)$

$L_c(n)$ $\quad (\mu=\emptyset,\ \nu=(1))$

② $n=2m$ $\quad L_c((m-1,^2 2))$,

$L_c(2m-1,1)$, $\quad \mu=(m-1,1),\ \nu=(1)$

$L_c(n)$, $L_c(m,m)$

Note eu acts semisimply on $B\otimes_{H_c} M$. The image is, in general, not closed/extension

④ Functors

$\mathcal{C}_c := \Phi_c(HC(c,c))$ If $W \subseteq S_n$ is a parabolic subgroup,

$\mathcal{C}_c(W) = $ image of Φ_c

Lemma (Losev) $\mathrm{Res}^{S_n}_W(\mathcal{C}_c) \subseteq \mathcal{C}_c(W)$. So we have induced restriction functors yogl

<u>Note</u> The inclusion $\iota : \mathcal{C}_c \to \mathcal{O}_c$ admits both a left and a right adjoint

$\iota^! : \mathcal{O}_c \to \mathcal{C}_c$ ($M \mapsto$ largest subobject of M in $\mathcal{C}_c$)

${}^!\iota : \mathcal{O}_c \to \mathcal{C}_c$ ($M \mapsto$ largest quotient of M in $\mathcal{C}_c$)

$$\begin{tikzcd}
& & \mathcal{C}_c \\
\mathcal{C}_c(W) \arrow[r, "\mathrm{Ind}_W^{S_n}"] \arrow[urr, bend left, "{}^{HC}\mathrm{CoInd}"] \arrow[drr, bend right, "\mathrm{Ind}"'] & \mathcal{O}_c \arrow[ur, "\iota^!"] \arrow[dr, "{}^!\iota"'] & \\
& & \mathcal{C}_c
\end{tikzcd}$$

${}^{HC}\mathrm{CoInd}(H_c(W)) = H_c$, ${}^{HC}\mathrm{Ind}(H_c(W)) = D_c$

<u>Duality</u> At the geometric level, this is Verdier duality
Need to be careful at the algebraic level

$$\mathcal{A}_\lambda(n,r) \xrightarrow{\cong} \mathcal{A}_\lambda(n,r)^{opp}$$

Not as quantizations! In the Hilbert scheme case, the ass. graded map induces the iso that interchanges x & y coordinates in $\mathbb{C}^2$.

Can use this iso to ~~...~~ & dg to geo.

$$\mathbb{D}: HC(c,c) \xrightarrow{\cong} HC(-c+N, -c+N)^{opp.}$$

$c = 1/m$

⑤ Principal block: block of $H_c(q,c)$ containing the regular bimodule

__Thm__ (Losev '12, Etingof-Stoica '09) The ideals of H_c form a chain

$$0 = J_{-1} \subsetneq J_0 \subsetneq \cdots \subsetneq J_{\lfloor n/m \rfloor -1} \subsetneq J_{\lfloor n/m \rfloor} = H_c$$

$$S_i := J_i / J_{i-1}, \quad i = 0, \dots, \lfloor n/m \rfloor$$

Under the ~~correspondence~~ functor, Φ_c, S_i corresponds to

$$L_c(m\text{-triv}(n-im) + (im))$$

__Thm__ (S.) The principal block is equivalent to the cat-y of reps of the quiver

$$\begin{tikzcd} S_0 \arrow[r, bend right, "\beta_1"'] & S_1 \arrow[l, bend right, "\alpha_1"'] \arrow[r, bend right, "\beta_2"'] & \cdots \arrow[l, bend right, "\alpha_2"'] \arrow[r, bend right] & S_{\lfloor n/m \rfloor -1} \arrow[l, bend right] \arrow[r, bend right, "\beta_{\lfloor n/m \rfloor}"'] & S_{\lfloor n/m \rfloor} \arrow[l, bend right, "\alpha_{\lfloor n/m \rfloor}"'] \end{tikzcd}$$

w/ relations $\alpha_i \beta_i = \beta_i \alpha_i = 0 \ \forall i$

duality interchanges α's & β's and fixes S_i's

Conj $\mathcal{C}_c$ is the Serre span of the simples it contains in

$$\mathcal{O}_c^{ev-s/s} \quad \text{(true for principal block)}$$

Q For which simples $L_c(\lambda)$ we have $\mathrm{Res}_W^{S_n} L_c(\lambda) \in \mathcal{O}_c^{ev-s/s}$