

HC bimodular for quantized Hilbert schemes

1) Quantized Hilbert schemes

Type A rational Cherednik algebras (type gl_n)
 $c \in \mathbb{C}$

[Etingof-Ginzburg, ca 2000] $H_c := \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \rtimes S_n$ / $[x_i, x_j] = 0, [y_i, y_j] = 0$
 $[y_i, x_j] = c(\delta_{ij})$
 $[y_i, x_i] = 1 - c \sum_{j \neq i} (\delta_{ij})$
 appeared in Minh-Tran Trinh

eg $c=0, H_0 = D(\mathbb{C}^n) \rtimes S_n$

$\mathbb{C}S_n \subseteq H_c, e = \frac{1}{n!} \sum_{w \in S_n} w$ idemp in H_c

et H_c is almost the algebra $A(n, 1)$ that appeared in I. Losev's talk

- filtered, $F_0 H_c = \mathbb{C}W, F_1 H_c = \mathbb{C}\{x_i^w, y_i^w \mid i=1, \dots, n, w \in W\}$

$F_k H_c = (F_1 H_c)^{\otimes k}$

$gr^F H_c = \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \rtimes S_n$

center $Z = \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle^{S_n}$

- graded, $\deg x_i = 1, \deg y_i = -1, \deg w = 0$

inner:

$eu = \frac{1}{2} \sum x_i y_i + y_i x_i$

Rep theory of \mathcal{H}_c has been focused on a caty of highest weight modules

\mathcal{O}_c [Jimbo-Guy-Ogata-Lusztig]

- highest weight caty

$$\begin{array}{ccccc} P_c(\lambda) & \twoheadrightarrow & \Delta_c(\lambda) & \twoheadrightarrow & L_c(\lambda) & \lambda \in \text{Irrep } S_n = \text{Partition} \\ \text{proj-ve} & & \text{standard} & & \text{simple} & \end{array}$$

Proj ev acts semisimply on $\Delta_c(\lambda), L_c(\lambda)$

[Parquet, Losev] $\mathcal{O}_c \cong S_{\mathfrak{g}(n)}\text{-mod} + \mathfrak{g}$ -Schur algebra $\mathfrak{g} = \exp(2\pi i F_1)$

2) HC bimodules

$$c, c' \in \mathbb{G} \quad g^r H_c = g^{r'} H_{c'}$$

Berest et al.
Ginzburg '05
Losev '12]

Def An $(H_c, H_{c'})$ -bimodule B is called HC if it admits a filtration \mathcal{F}

- $g^r B$ is a $\text{fg } \mathbb{C}\langle X, Y \rangle \rtimes S_n$ bimodule
- the left and right actions of \mathbb{Z} on $g^r B$ coincide

$$SS(B) := \text{supp}(g^r B) \subseteq \text{Sym}^n \mathbb{G}^2$$

totally analogous
to HC $\mathbb{C}\langle X, Y \rangle$ bimodules

Example 1) H_c is a HC H_c -bimodule

2) Subquotients (cong) and extensions (hull) of HC bimodules are HC.

3) Translation bimodules are HC

$HC(c, c')$ - caty of HC bimodules (\otimes caty if $c=c'$)

Want Describe this caty

Note equivalent to caty of
modules / f.d algebra

$$\underline{Prop}$$
 If $B \in HC(c, c'), M \in \mathcal{O}_{c'} \Rightarrow B \otimes_{H_c} M \in \mathcal{O}_c$

Thm (S.) $HC(c, c') = 0$ unless

1) $c - c' \in \mathbb{Z}$ or $c + c' \in \mathbb{Z}$, or

2) $c = \frac{r}{m}, c' = \frac{r'}{m}, \text{gcd}(r, m) = \text{gcd}(r', m) = 1$, and $m | r$.

In case 1, $D_{HC}((H_c, H_{c'}) \text{ bimod}) \cong D_{HC}(H_c \text{-bimod})$

In case 2, but not 1, $HC(c, c') \cong S_{\frac{r}{m}}\text{-rep}$

③ HC vs. GGOR

$$\Phi_c: HC(c, c) \rightarrow \mathcal{O}_c$$

$$B \mapsto B \otimes_{\mathbb{H}} \Delta(\text{triv})$$

see $\Delta(\text{sign})$ for $c < 0$.

Type A only!

Thm (S) Φ_c is a ^{exact} fully faithful embedding whose image is closed under subquotients. [Analogue to Bernstein-Gelfand thm for enveloping algebras]

- Simplex in the image of Φ_c :

Mention ~~right~~ ~~inv~~ ~~left~~!

$c \in \{ \frac{r}{m} \mid 1 \leq m \leq n, \text{gcd}(r, m) = 1 \} \Rightarrow$ the only simple in the image of Φ_c is $L(\text{triv}) = \Delta(\text{triv})$
 $\Rightarrow HC(c, c) \cong^{\otimes} \text{Vect}$

very boring

$c \in \mathbb{Z} \Rightarrow \Phi_c$ is an equivalence [BEG], $HC(c, c) \cong^{\otimes} S_{n-1}\text{-rep}$

box part of
 \mathbb{Z} function
 Wilcox,
 Losev-Shelley
 Ardhanari

$c = \frac{r}{m}, 1 < m \leq n, \text{gcd}(r, m) = 1$. $L_c(\lambda) \in \text{Im } \Phi_c$ iff in the dec $\lambda = \mu + m\epsilon$, μ is m -trivial
 $\lambda + n \Rightarrow \exists!$ decomposition $\lambda = \mu + m\epsilon$, μ is m -regular ($\mu_i - \mu_{i-1} < m$)

A partition μ is called m -trivial iff $\mu = ((m-u)^a, b)$, $a \geq 0$, $0 \leq b < m-1$

$$H_c = \text{Hom}_{\mathfrak{g}_n}(\Delta_c(\text{triv}), \Delta_c(\text{triv}))$$

$$D_c = \text{Hom}_{\mathfrak{g}_n}(\Delta_c(\text{triv}), \nabla_c(\text{triv}))$$

double wall-crossing bimodule [Bezrukavnikov-Losev]

Example 1 $n = m$ $L_c((n-1, 1))$ ($\mu = (n-1, 1)$, $\nu = \emptyset$)

$$L_c(n) = (\mu = \emptyset, \nu = (1))$$

Example 2 $n = 2m$ $L_c((m-1, 2))$,

$$L_c(2m-1, 1), \quad \mu = (m-1, 1), \quad \nu = (1)$$

$$L_c(n), L_c(m, 1)$$

Note e_i acts semisimply on $\mathbb{S} \otimes_{\mathbb{K}} M$. The image is, in general, not closed/extension

④ Functor

$\mathcal{P}_C := \Phi_C(\text{HC}(C, C))$ If $W \subseteq S_n$ is a parabolic subgroup,

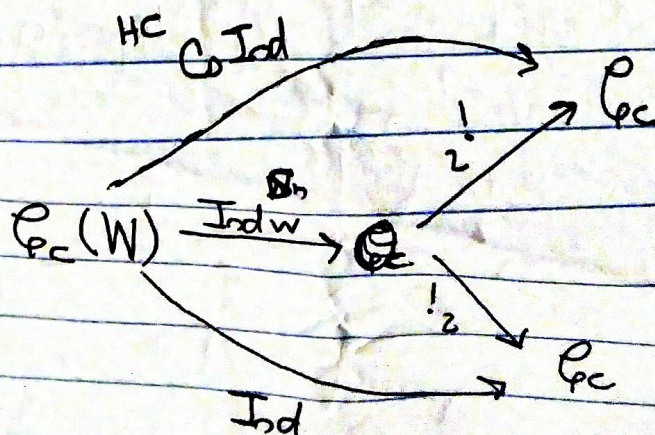
$$\mathcal{P}_C(W) = \text{image of } \Phi_C$$

Lemma (Lusztig) $\text{Res}_W^S(\mathcal{P}_C) \subseteq \mathcal{P}_C(W)$. So we have induced restriction functor ygl

Note The inclusion $z: \mathcal{P}_C \rightarrow \mathcal{O}_C$ admits both a left and a right adjoint

$$z': \mathcal{O}_C \rightarrow \mathcal{P}_C \quad (M \mapsto \text{largest subobject of } M \text{ in } \mathcal{P}_C)$$

$$!z: \mathcal{O}_C \rightarrow \mathcal{P}_C \quad (M \mapsto \text{largest quotient of } M \text{ in } \mathcal{P}_C)$$



$$\text{HC Co-Ind}(\text{HC}(W)) = \text{HC}, \quad \text{HC Ind}(\text{HC}(W)) = \mathcal{D}_C$$

Duality At the geometric level, this is Verdier duality
Need to be careful at the algebraic level

$$D_2(n, r) \xrightarrow{\cong} D_2(n, r)^{opp}$$

Not a quantization! In the Hilbert scheme case,
the ass. graded map induces the
iso that interchanges x & y
coordinates in \mathbb{G}^2 .

Can we then go to ~~the~~ d_2 to go.

$$D: HC(c, d) \xrightarrow{\cong} HC(-c+N, -c+N)^{opp}$$

$$c = \ell/m$$

⑤ Principal block: Block of $HC(c, \ell)$ containing the regular bimodule

Thm (Losev '12, Etingof-Sturka '09) The ideals of H_c form a chain

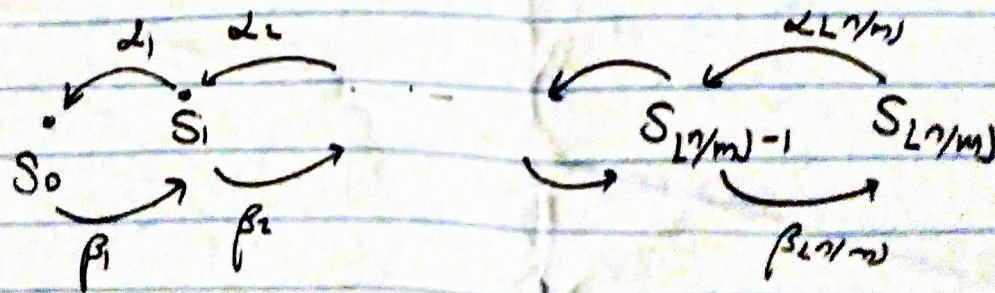
$$0 = J_{-1} \subsetneq J_0 \subsetneq \dots \subsetneq J_{\lfloor \ell/m \rfloor - 1} \subsetneq J_{\lfloor \ell/m \rfloor} = H_c$$

$$S_i := J_i / J_{i-1}, \quad i = 0, \dots, \lfloor \ell/m \rfloor$$

Under the ~~isomorphism~~ functor Φ_c , S_i corresponds to

$$L_c(m - \text{triv}(\ell - im) + (im))$$

Thm (S.) The principal block is equivalent to the cat-y of repr of the quiver



w/ relations $\alpha_i \beta_i = \beta_i \alpha_i = 0 \quad \forall i$

duality interchanges α 's & β 's and fixes S_i 's

Case \mathcal{O}_C is the Serre space of the simplex it contains in

$$\mathcal{O}_C^{eu-s/s} \quad (\text{true for principal block})$$

Q For which simplex $L_C(\mathcal{R})$ we have $\text{Per}_W^S L_C(\mathcal{R}) \in \mathcal{O}_C^{eu-s}$