Virtualization of root systems and Littelmann Path Model

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Overview

Root Systems Basics

Reflections

Let V be a Euclidean space. If $0 \neq \alpha \in V$, then r_α is the reflection in the hyperplane orthogonal to α.

(Reduced) Root Systems

A root system Φ in V is a nonempty finite set of nonzero vectors in V such that

1. $r_\alpha(\Phi) = \Phi \quad \forall \alpha \in \Phi$;
2. $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z} \quad \forall \alpha, \beta \in \Phi$;
3. if $\beta \in \Phi$ is a multiple of $\alpha \in \Phi$, then $\beta = \pm\alpha$.

Coroots

$$\alpha^\vee := \frac{2\alpha}{(\alpha, \alpha)}$$

Finite Root Systems

- 4 infinite families: A_n, B_n, C_n, D_n.
- 5 exceptional types: $E_{6,7,8}$, F_4, G_2.
- Information can be recorded in Dynkin diagrams.

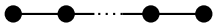


Figure: Type A_n

Figure: Type B_n

Figure: Type C_n

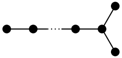


Figure: Type D_n

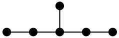


Figure: Type E_6

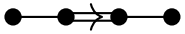
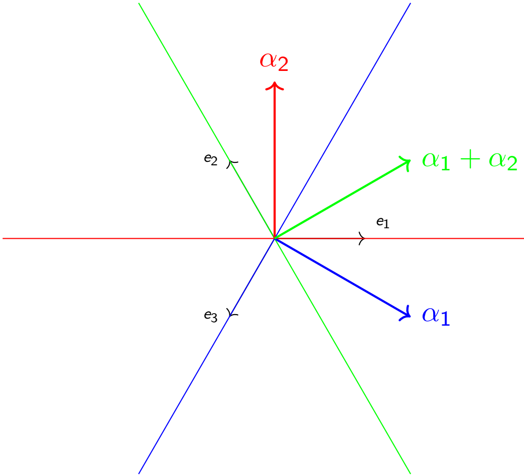


Figure: Type F_4

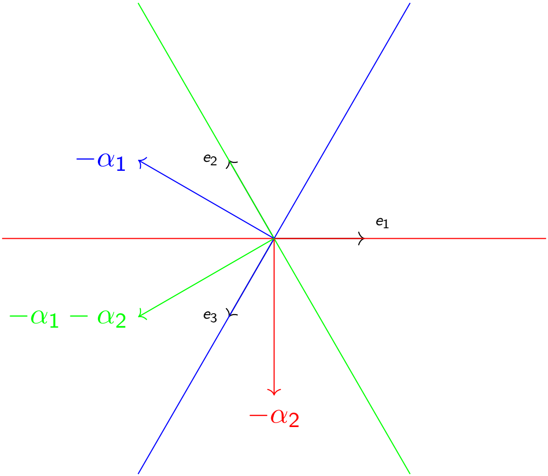
Figure: Type G_2

Type A_2



- $e_1 + e_2 + e_3 = 0$
- $(e_i, e_j) = \delta_{ij}$
- $\alpha_1 = e_1 - e_2 = \alpha_1^\vee$
- $\alpha_2 = e_2 - e_3 = \alpha_2^\vee$
- Simply-laced

Type A_2



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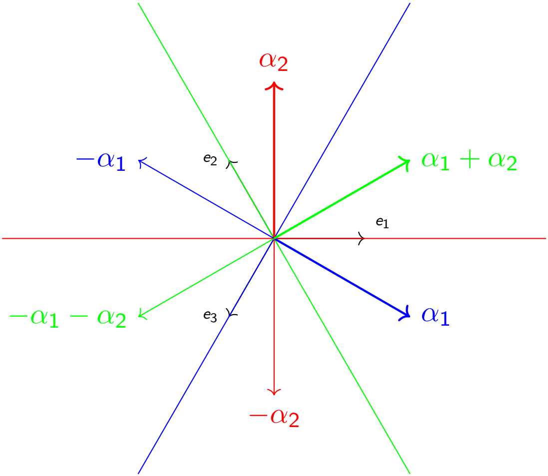

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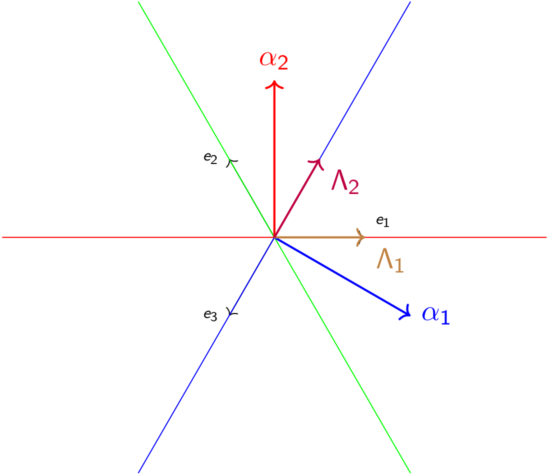
Fundamental Weights

- The fundamental weights $\{\Lambda_i\}_i$ are defined as the dual basis of $\{\alpha_i^\vee\}_i$ with respect to $\langle\ ,\ \rangle$, *i.e.*, $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$.
- The weight lattice is $\Lambda := \bigoplus_i \mathbb{Z}\Lambda_i$.
- The dominant chamber $\Lambda^+ := \bigoplus \mathbb{R}_{\geq 0}\Lambda_i$ corresponds to the identity element.

Example

In type A_n, we have $\Lambda_i = e_1 + e_2 + \cdots + e_i$ for $1 \leq i \leq n$.

Example: A_2 (again)



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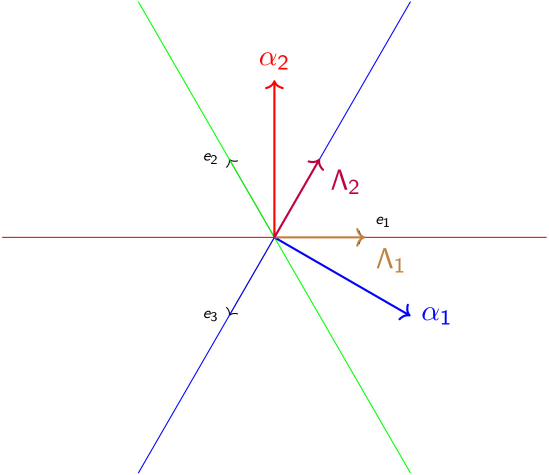

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- $\Lambda_2 = e_1 + e_2$
- $\langle \Lambda_1, \alpha_1^\vee \rangle = 1$
- $\langle \Lambda_1, \alpha_2^\vee \rangle = 0$
- $\langle \Lambda_2, \alpha_1^\vee \rangle = 0$
- $\langle \Lambda_2, \alpha_2^\vee \rangle = 1$

Kashiwara Crystal

Definition

Fix a root system Φ with index set I and weight lattice Λ. A Kashiwara crystal of type Φ is a nonempty set $\mathcal{B}$ together with maps

$$e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$$
$$\mathrm{wt} : \mathcal{B} \to \Lambda$$

where $e_i x = y$ if and only if $f_i y = x$. Together with some other axioms.

Representation Theoretic Motivation for Crystals

- Irreducible representations V_λ and V_μ
- $$V_\lambda \otimes V_\mu \cong \bigoplus_\nu c_{\lambda\mu}^\nu V_\nu$$

Question: How to count multiplicities $c_{\lambda\mu}^\nu$?

- Crystals $\mathcal{B}_\lambda \longleftrightarrow V_\lambda$, $\mathcal{B}_\mu \longleftrightarrow V_\mu$
- $$\mathcal{B}_\lambda \otimes \mathcal{B}_\mu = \bigoplus_\nu c_{\lambda\mu}^\nu \mathcal{B}_\nu$$
- (Type A)
  $c_{\lambda\mu}^\nu = \#\{\text{Yamanouchi tableaux of shape } \nu/\lambda \text{ and content } \mu\}$
  Littlewood-Richardson Coefficients

Example of type A_2

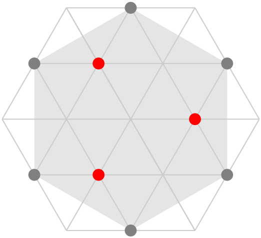

Figure: Std Rep of $\mathfrak{sl}_3$: $V_{(1,0)}$

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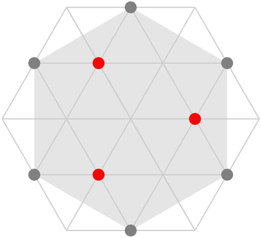

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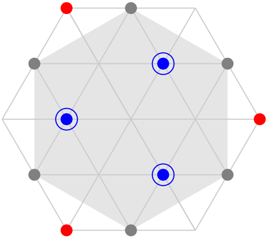

Figure: Tensor Product

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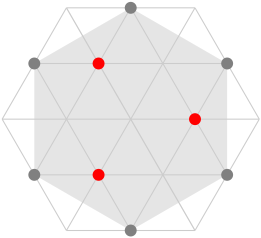

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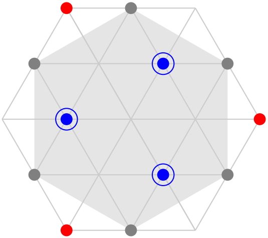


Figure: Tensor Product

$$1 \xrightarrow{\;1\;} 2 \xrightarrow{\;2\;} 3$$

$$\begin{array}{cccccccc}
1 & & 1\otimes 1 & \xrightarrow{1} & 1\otimes 2 & \xrightarrow{2} & 1\otimes 3 \\
\downarrow{\scriptstyle 1} & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
2 & & 2\otimes 1 & & 2\otimes 2 & \xrightarrow{2} & 2\otimes 3 \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle 2} \\
3 & & 3\otimes 1 & \xrightarrow{1} & 3\otimes 2 & & 3\otimes 3
\end{array}$$

Nice Things about Simply-laced Types

Theorem (Stembridge, 03')

Assume that the root system is simply-laced. Let $\mathcal{C}$ be a connected weak Stembridge crystal that is nonempty, upper seminormal and bounded above. Then $\mathcal{C}$ has a unique highest weight element.

Theorem (Stembridge, 03')

Let $\mathcal{C}$ and $\mathcal{C}'$ be connected Stembridge crystals. If their highest weight elements have the same weight, then they are isomorphic.

Virtualization map

Virtualization [Kashiwara 96']

Consider root systems Φ (resp. $\widehat{\Phi}$) with index sets I (resp. $\widehat{I}$) simple roots $\{\alpha_i\}_i$ (resp. $\{\widehat{\alpha}_i\}_i$) and fundamental weights $\{\Lambda_i\}_i$ (resp. $\{\widehat{\Lambda}_i\}$).

A virtualization of Φ by $\widehat{\Phi}$ with folding $\phi \colon \widehat{I} \to I$ and scaling factors $\{\gamma_i\}_i$ is a linear map

$$\Lambda_i \mapsto \gamma_i \sum_{j \in \phi^{-1}(i)} \widehat{\Lambda}_j$$

such that

- $\langle \widehat{\alpha}_j, \widehat{\alpha}_{j'} \rangle = 0$ for all $j, j' \in \phi^{-1}(i)$;
- $$\alpha_i \mapsto \gamma_i \sum_{j \in \phi^{-1}(i)} \widehat{\alpha}_j.$$

Some Natural Virtualization

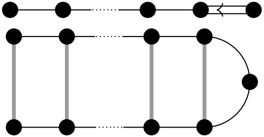


Figure: $C_n \hookrightarrow A_{2n-1}$

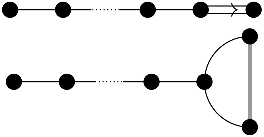


Figure: $B_n \hookrightarrow D_{n+1}$

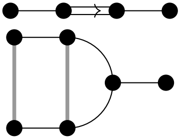


Figure: $F_4 \hookrightarrow E_6$

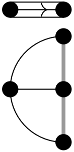


Figure: $G_2 \hookrightarrow D_4$

Virtual crystals

Virtual Crystals [Kashiwara 96']

Consider a virtualization of the root system Φ to $\widehat{\Phi}$ where v is the map on the weight lattices. Let $\widehat{\lambda} = v(\lambda)$. We say $B(\lambda)$ is a virtual crystal of $B(\widehat{\lambda})$ if there exists a subset V of $B(\widehat{\lambda})$ that is isomorphic to $B(\lambda)$ under the crystal structure

$$e_i := \prod_{j \in \phi^{-1}(i)} \widehat{e}_j^{\gamma}, \quad f_i := \prod_{j \in \phi^{-1}(i)} \widehat{f}_j^{\gamma}, \quad v \circ \mathrm{wt} = \widehat{\mathrm{wt}}.$$

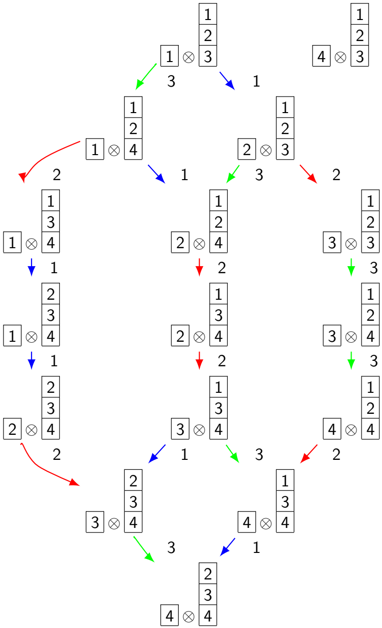
We call the resulting isomorphism $\Psi \colon B(\lambda) \to V$ the virtualization map.

Nice things about Virtual Crystals

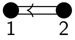
Theorem (Bump, Schilling 16')
Let $\mathcal{V} \subseteq \widehat{\mathcal{V}}$ be a connected virtual crystal for the Lie algebra embedding $X \hookrightarrow Y$. Then $\mathcal{V}$ has a unique highest weight element.

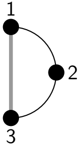
Theorem (Bump, Schilling 16')
Let $\mathcal{V}, \mathcal{V}' \subseteq \widehat{\mathcal{V}}$ be connected virtual crystals corresponding to the Lie algebra embedding $X \hookrightarrow Y$. If their highest weight elements have the same weight, then they are isomorphic.

Virtualize C_2 in A_3



Example

- 

- 

- $\Lambda_1 \mapsto \widehat{\Lambda}_1 + \widehat{\Lambda}_3$
- $\Lambda_2 \mapsto 2\widehat{\Lambda}_2$

- $f_1 \mapsto \widehat{f}_1 \widehat{f}_3$
- $f_2 \mapsto \widehat{f}_2^{\,2}$

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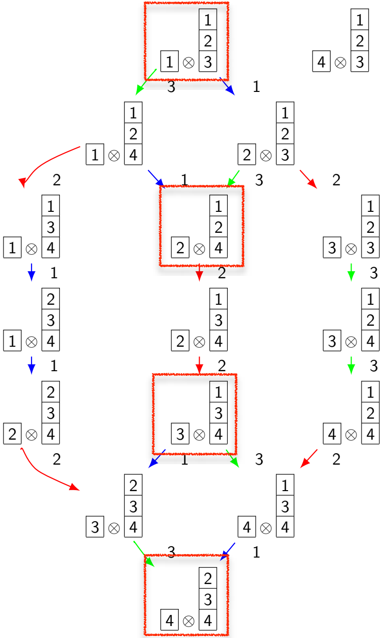

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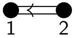
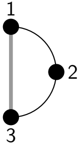
- 

- 

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- $f_1 \mapsto \widehat{f}_1 \widehat{f}_3$
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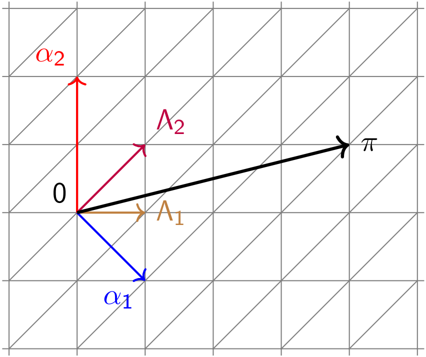
Littelmann path model

- Paths $\pi\colon [0,1] \to \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ up to reparameterization.
- $\pi(0) = 0,\ \pi(1) \in \Lambda$.
- The closure under f_i from the straight-line path $u_\lambda(t) = \lambda t$ is the irreducible extremal weight crystal $B(\lambda)$. When λ is dominant, $B(\lambda)$ is a highest weight crystal.

Example on C_2



- $\pi(t) = (3\Lambda_1 + \Lambda_2)t$

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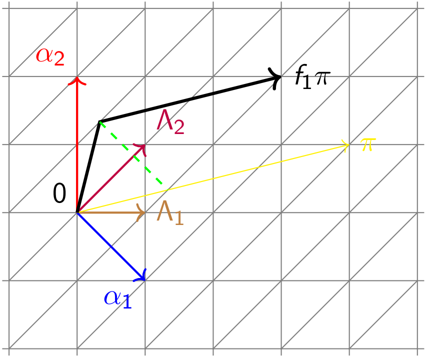
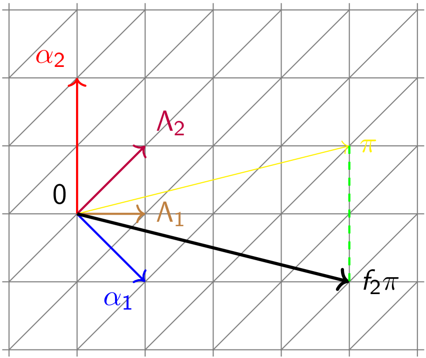


- $\pi(t) = (3\Lambda_1 + \Lambda_2)t$
- $\langle (3\Lambda_1 + \Lambda_2)t, \alpha_1^\vee \rangle = 3t$
- Largest $t \in [0, 1]$ attains the minimum: $t = 0$
- Minimal $t' \in [t, 1]$ such that $3t' = 1$: $t' = 1/3$

Example on C_2



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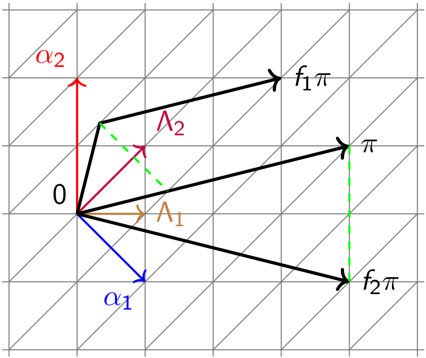
- $\langle (3\Lambda_1 + \Lambda_2)t, \alpha_2^\vee \rangle = t$
- Largest $t \in [0, 1]$ attains the minimum (0): $t = 0$
- Minimal $t' \in [t, 1]$ such that $t' = 1$: $t' = 1$

Example on C_2



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Results

Theorem (P-Scrimshaw)

Let Φ *to* $\widehat{\Phi}$ *be root systems with weight lattices* Λ *and* $\widehat{\Lambda}$ *respectively.*

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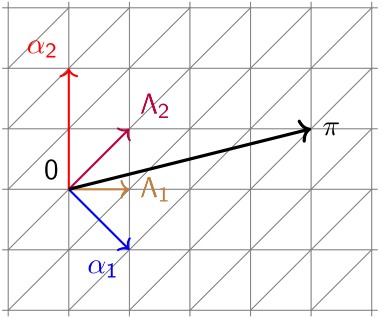
Theorem (P-Scrimshaw)

Let Φ to $\widehat{\Phi}$ be root systems with weight lattices Λ and $\widehat{\Lambda}$ respectively.

The following are equivalent:

- *There exists a virtualization of Φ to $\widehat{\Phi}$.*
- *The embedding of weight lattices $v : \Lambda \to \widehat{\Lambda}$ is a virtualization map on the Littelmann path model.*
- *There is a virtualization of crystals $B(\lambda)$ to $B(\widehat{\lambda})$.*

Example



- $\pi(t) = (3\Lambda_1 + \Lambda_2)t$
- $f_2(\pi)(t) = (3\Lambda_1 - \Lambda_2)t$
- $\tilde{\Psi}(\pi)(t) = (3\hat{\Lambda}_1 + 2\hat{\Lambda}_2 + 3\hat{\Lambda}_3)t$
- $\tilde{\Psi}(f_2\pi)(t) = (5\hat{\Lambda}_1 - 2\hat{\Lambda}_2 + 5\hat{\Lambda}_3)t$
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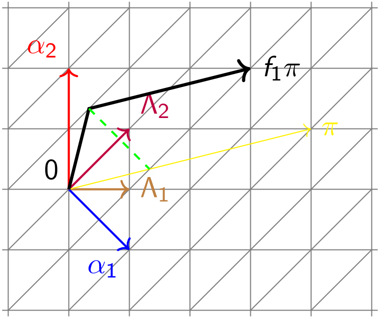

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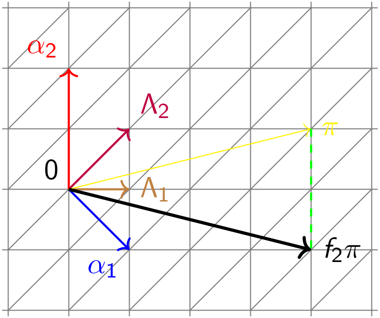

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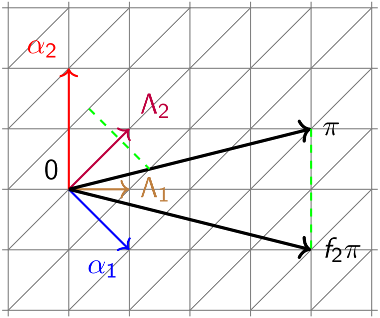

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Results

Proposition

Let $\mathfrak{g}$ be of affine type. Let $\widetilde{\Psi}$ be the virtualization map induced from the generalized diagram folding. Then there exists a $U_q'(\mathfrak{g})$-crystal virtualization map $\widetilde{\Psi}_{\mathrm{cl}}$ such that the diagram

$$\begin{array}{ccc} B(\lambda) & \xrightarrow{\quad \widetilde{\Psi} \quad} & B\bigl(\Psi(\lambda)\bigr) \\ {\scriptstyle \mathrm{cl}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{cl}} \\ B(\lambda)_{\mathrm{cl}} & \xrightarrow[\quad \widetilde{\Psi}_{\mathrm{cl}} \quad]{} & B\bigl(\Psi(\lambda)\bigr)_{\mathrm{cl}} \end{array}$$

commutes.

Conjecture

Conjecture

The KR crystal $B^{r,s}$ of type $\mathfrak{g}$ virtualizes into

$$\widehat{B}^{r,s} = \begin{cases} B^{n,s} \otimes B^{n,s} & \text{if } \mathfrak{g} = A_{2n}^{(2)}, A_{2n}^{(2)\dagger} \text{ and } r = n, \\ \bigotimes_{r' \in \phi^{-1}(r)} B^{r',\overline{\gamma}_r s} & \text{otherwise.} \end{cases}$$

Theorem

Let $\mathfrak{g}$ be of affine type. Suppose $r \in I$ is such that $\overline{\gamma}_r = 1$ or $\mathfrak{g}$ is of type $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$. Then the conjecture above holds for $s = 1$.

Thank you!