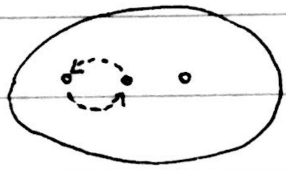
Licata, Braids, triangulated autoeq's, rep thy.

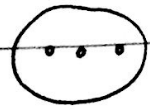
I. Σ : (orientable) surface

$\Gamma(\Sigma)$: $\mathrm{Diffeo}^+(\Sigma)/_{\text{isotopy}}$ ← mapping class group

$\Gamma \curvearrowright \{$simple closed (multi)curves on $\Sigma\}$.

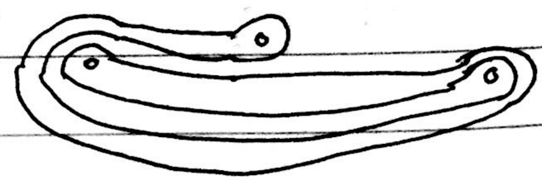
Ex. Σ =  , $\Gamma = Br_3$.

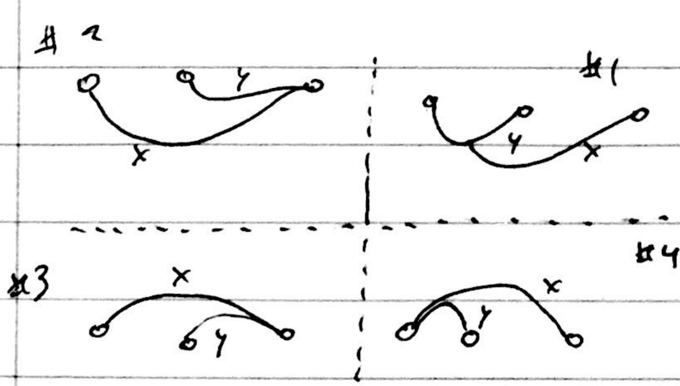
Q. Given $g \in \Gamma(\Sigma)$: how do curves on Σ "grow" under repeated application of g?

"Train tracks" give a notation for curves on 

Ex.

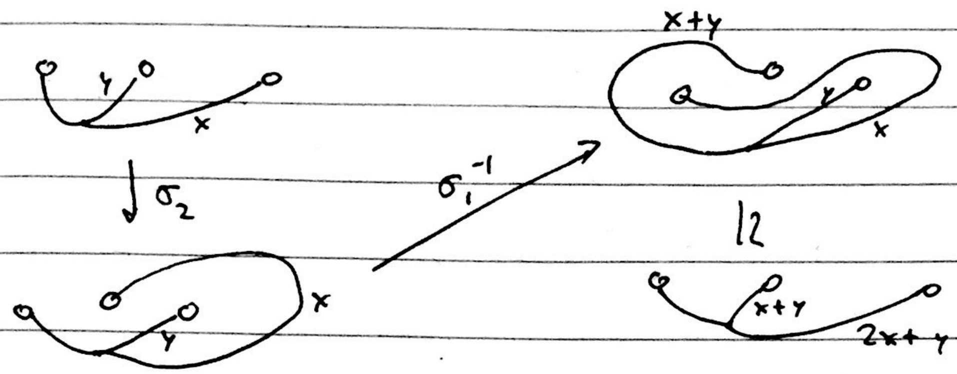
means:





Thurston: $\sigma_1^{-1}\sigma_2 \curvearrowright$ curves in the fixed chart #1

"charts" for set of multicurves



so $\sigma_1^{-1}\sigma_2$

$\updownarrow$

$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

<u>on this chart</u>

II. $\Gamma(\Sigma) \circlearrowright \text{Teich}(\Sigma) = \left\{ \begin{matrix} \text{hyperbolic } \text{metrics} \\ \text{in } \Sigma \end{matrix} \right\} / \sim$

$\cong$ — use F.N. coords?

open ball

$\text{Teich} \subseteq \overline{\text{Teich}} = \text{Teich} \sqcup \text{PMF}$

$\overline{\text{Teich}} \cong$ closed ball

PMF: "projective measured foliations" $\cong$ sphere

any simple closed curve gives a PMF.

$S = \{\text{simple closed multicurves on } \Sigma\}/\sim$

$\text{Teich} \longrightarrow \mathbb{R}^S_{\geq 0} = \{f : S \to \mathbb{R}\}$.

$\gamma \longmapsto (c \mapsto l_\gamma(c))$.

$S \longrightarrow \mathbb{R}^S_{\geq 0}$

$c \longmapsto (c' \mapsto |\text{intersection } c \cdot c'|)$

Thurston cpt'ification: $\text{Teich} \xhookrightarrow{j} \mathbb{P}(\mathbb{R}^S_+)$

$S \xrightarrow{k}$

$\Rightarrow \overline{\text{im}(j)} = \underbrace{\text{im}(j)}_{\text{Teich}} \cup \underbrace{\overline{\text{im}(k)}}_{\text{PMF}}$

$\overline{\text{Teich}} = \text{Teich} \amalg \text{PMF}$

Properties: 1. $\overline{\text{Teich}}$ maps homeo'lly onto its image in $\mathbb{P}(\mathbb{R}^S_+)$

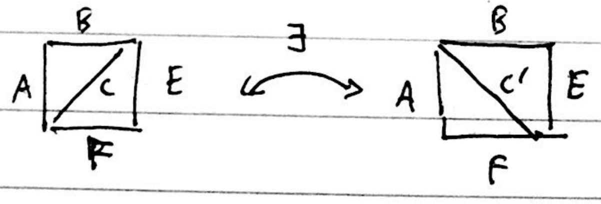
2. $\overline{\text{Teich}}$ = closed. f.d. Euclidean ball

3. $\Gamma(\Sigma)$ acts <u>piece-wise</u> <u>linearly</u> on the boundary, PMF.

"Triangulated cats are two-dim objects." (!?)

(~~Ka~~ Dyckeroff - Kapranov)

<u>Ex</u>. the "octahedral" axiom "says"



(!!)

III. Fix triangulated cat T.

Analogy:

Σ	T
$\Gamma(\Sigma)$	Aut(T)
Teich	$\underbrace{\text{Stab(T)}}_{\text{moduli of Bridgeland stability conditions}}$
S (in a chart?)	semistable objs for $\sigma \in$ Stab

Stability conditions: $\sigma \in \mathrm{Stab}(T)$ gives:

1. set of (semi-)stable objects $\{E_\alpha\}_\alpha$ in T.

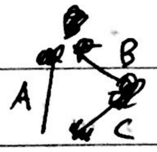
2. for each E_α, mass $m(E_\alpha) \in \mathbb{R}_{>0}$
   phase $\phi(E_\alpha) \in \mathbb{R}$

will set $m_t(E_\alpha) = m(E_\alpha)\, e^{\pi i t \phi(E_\alpha)} \in \mathbb{C}$.

Such that every $X \in T$ has canonical H-N filt

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{n-1} \arrow[r] \arrow[d] & X_n = X \arrow[d] \\
& E_1 & E_2 \arrow[ul, dashed] & & E_{n-1} \arrow[ul, dashed] & E_n \arrow[ul, dashed]
\end{tikzcd}$$

E_1, E_2, E_{n-1}, E_n } semistable

Implicit: 

$\Rightarrow$ $m_t(A) \leq m_t(B) + m_t(C)$.

$m(X) = \sum_i m(E_i)$

"A stability condition is a way to measure obj's in a triangulated category."

Thm (Bridgeland). $\mathrm{Stab}(T)$ is a complex manifold.
In all ex's computed, it's contractible!

Ex. If T = 2-Calabi-Yau cat of an ADE quiver then

$\mathrm{Stab}(T)$ is a ~~[illegible]~~ Euclidean ball. (dim = rk of Groth group)

Teich itself is a Stab($\mathcal{T}$).

(something about Fukaya cats)

Why useful to compactify Teich?
Better to study actions on a closed ball (ie. cpt)
than an open ball.
(Brouwer/ Lefschetz ... ??)

So want to "compactify" Stab($\mathcal{T}$). Copy Thurston.

related to Thurston-Nielsen classif.

$$\mathrm{Stab}(\mathcal{T}) \longrightarrow \mathbb{R}^S_{>0} \longrightarrow \mathbb{P}(\mathbb{R}^S_{>0}).$$

$$\sigma \longmapsto \begin{pmatrix} f_\sigma : S \to \mathbb{R} \\ x \mapsto m_t^\sigma(x) \end{pmatrix}$$

where now, $S = \{$ semi-stable objects in $\mathcal{T}\}$.
for any stability cond.

joint with Bapat and Deopurkar:

Df. $\overline{\mathrm{Stab}(\mathcal{T})} := \overline{\mathrm{im}(\mathrm{Stab}(\mathcal{T}))} \subseteq \mathbb{P}(\mathbb{R}^S_{>0})$.

Conj. At least for CY cats of ADE quivers,

$\overline{\mathrm{Stab}(\mathcal{T})}$ = closed ball

$\left(\begin{array}{c} \| \\ \mathrm{Stab}(\mathcal{T}) \sqcup \text{sphere} \end{array} \right)$

What is the sphere ??

In the case of 2-CY of ADE:

$$\begin{aligned} \big(&= D^b(\mathrm{Rep}_{\mathrm{PreProj}}(Q)) \\ &= K^b(\mathrm{Mod}_{\mathrm{ZigZag}}(Q)) \\ &= \underset{?}{\text{quotient}} \text{ of } K^b(\mathrm{SBim}) \end{aligned}$$

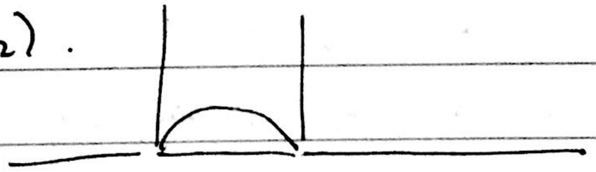
$S = \{ \text{spherical obj's} \}$

$$S \longrightarrow \mathbb{P}(\mathbb{R}^S)$$

$$E \longmapsto \left(\begin{aligned} f_E : S &\to \mathbb{R} \\ Y &\mapsto \dim \mathrm{Hom}^*(E, Y) \end{aligned} \right)$$

Prop. S lives in the sphere boundary.
when $t=0$, it's dense.

Ex. (type A_2).
$t=0$:



bdy is $\mathbb{Q}$

$S = \{P_1, P_2, \beta(P_1)\}$

$t \neq 0$: boundary some Cantor set.