<u>Losev</u>, Gieseker Moduli Spaces & Higher Rk Catalan #s

1. $r, n \in \mathbb{Z}_{\geq 0}$.

$R = \mathrm{End}(\mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)$.

$G = GL_n(\mathbb{C}) \circlearrowright R$.

$T^*R \simeq R \oplus R^*$ via trace form.

$$\begin{array}{ccc} T^*R & \ni & (A, B, i, j) \quad \text{s.t.} \quad A, B \in \mathrm{End}(\mathbb{C}^n). \\ \mu \big\downarrow & & i : \mathbb{C}^n \to \mathbb{C}^r \\ \mathfrak{g}^* & & j : \mathbb{C}^r \to \mathbb{C}^n. \end{array}$$

$\mu(A, B, i, j) = [A, B] - j \cdot i$.

$\mu^* : \mathfrak{g} \longrightarrow \mathbb{C}[T^*R]$.

$\xi \longmapsto \xi_R$, vec field on R.

Gieseker moduli space

$M(n, r) = \mu^{-1}(0) /\!/ G$.

$= \mathrm{Spec}\left[\mathbb{C}[T^*R] \big/ \langle \mu^*(\mathfrak{g}) \rangle \right]^G$.

affine, Poisson, ... singular

More generally, fix $\theta : G \to \mathbb{C}^\times$. Then have

$M^\theta(n, r) = \mu^{-1}(0)^{\theta\text{-ss}} /\!/_\theta G$. e.g. $\theta = \det$.

Ex. $M^{\det}(n, r) = \{ (A, B, i, j) :$ ker(i) has no A- or B-inv. subspaces $\} / G$.

"One [slash] or two [slashes]. There's something for everybody on this board."

$\mathcal{M}^{\theta}(n,r)$ ← smooth, irr, symplectic of dim $2nr$.

$\downarrow$ is a res. of sing.

$\mathcal{M}(n,r)$

"[On i,j conventions] I'm very sorry. I didn't meet yr expectations."

Replace $\mathrm{End}(\mathbb{C}^n)$ w $\mathfrak{sl}_n$.

dim $2nr \rightsquigarrow 2nr-2$.

Ex. $n=1$: $\mathcal{M}^{\theta}(1,r) = T^*\mathbb{P}^{r-1}$

$$\downarrow$$

$\mathcal{M}(1,r) = \overline{\left(\begin{array}{c}\text{min nilp orbit}\\ \text{in } \mathfrak{gl}_r\end{array}\right)}$

$r=1$: $\mathcal{M}^{\theta}(n,1)$ ← "basically" $\mathrm{Hilb}^n(\mathbb{C}^2)$.

$$\downarrow$$

$\mathcal{M}(n,1) = (\mathfrak{h} \oplus \mathfrak{h}^*) /\!/ S_n$ on $\mathbb{C}^n$

2. Quantization. $\lambda \in \mathbb{C}$.

$$A_\lambda(n,r) = \left[\frac{D(R)}{D(R)\cdot\{\xi_R - \lambda\cdot\mathrm{tr}(\xi) \mid \xi \in \mathfrak{g}\}}\right]^G.$$

assoc alg w/ filt induced by order filt. on $D(R)$.

Can show $\mathrm{gr}\, A_\lambda(n,r) = \mathbb{C}[\mathcal{M}(n,r)]$

so A_λ is a "quantization" of $\mathcal{M}(n,r)$.

(quantized Gieseker moduli space

Ex. $A_\lambda(n,1)$ is the spherical rat'l Cherednik alg for $S_n \curvearrowright \mathfrak{h}$.

$\mathbb{C}^\times \times GL_r \curvearrowright R$. commuting w/ G-action.

$z \times h \curvearrowright (A, i) \mapsto (zA, h \circ i)$.

so action descends to $\mathcal{M}(n,r)$,
$\mathcal{M}^\theta(n,r)$,
$A_\lambda(n,r)$.

"I just want to AWARE members of the audience of this fact."

First step toward rep thy: Describe f.d. reps

3 Thm 1 (Bezrukavnikov - Losev).

1. $\exists$ f.d. ~~rep~~ rep. of $A_\lambda(n,r)$ if and only if

$$\begin{cases} a)\ \lambda = \frac{a}{n} \text{ w/ } \gcd(a,n)=1. \\ b)\ \lambda \notin (-r, 0) \end{cases}$$

2. If $\exists$ f.d. then cat of such reps.

$\cong$

f.d. v.s.

Ex. $(n=1)$. $\mathcal{D}^\lambda(\mathbb{P}^{r-1})$ is a quotient of $U(\mathfrak{sl}_r)$.

$\curvearrowright$

$\Gamma(\mathcal{O}(\lambda))$ for $\lambda \geq 0$.

$(r=1)$. Berest - Etingof - Ginzburg.

For gen'l case: Need both rep thy of RCA
and geometry of $\mathcal{M}(n,r)$.

4. <u>Construction</u> of f.d. irrep.

Let $\underline{D(R)\text{-mod}}^{G,\lambda} := \left\{ (M,\rho) : \begin{array}{l} M \in \underline{D(R)\text{-mod}} \\ \rho : G \times M \to M \\ \text{st } \rho(\xi) = \xi_R - \lambda \cdot tr(\xi) \\ \text{for all } \xi \in \mathfrak{g} \end{array} \right\}$

$$M \in \underline{D(R)\text{-mod}}^{G,\lambda}$$
$$\downarrow$$
$$\underbrace{D(R)^G \twoheadrightarrow A_\lambda \circlearrowright M^G}$$

gives a $\underbrace{\text{quotient functor}}_{\exists \text{ right inverse}}$ $\underline{D(R)\text{-mod}}^{G,\lambda} \twoheadrightarrow \underline{A_\lambda\text{-mod}}$

Fact. $\exists!\ M_{a/n} \in \underline{D(\mathfrak{sl}_n)^{SL_n}\text{-mod}}$. st
$(a>0,\ gcd(a,n)=1)$

- $M_{a/n}$ irred.
- $supp(M_{a/n}) \subseteq$ nilp cone.

$\{(z,\dots,z) : z^n=1\} \subseteq SL_n$ acts on $M_{a/n}$, via z^{-a}.

Similarly $z \in GL_n \circlearrowright M_{a/n}$.

$$\begin{array}{ccc} M_{a/n} & \rightsquigarrow & M_{a/n} \boxtimes \mathbb{C}[\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)] \\ \cap & & \cap \\ \mathcal{D}(\mathfrak{sl}_n)\text{-mod}^{G,\lambda} & & \mathcal{D}(R)\text{-mod}^{G,\lambda} \end{array}$$

<u>Prop.</u> (Etingof - Krylov - L).

$$L_{a/n} = \left(M_{a/n} \boxtimes \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)\right)^G$$

is a f.d. A_λ-irrep.

Can compute its dim.

<u>5.</u> <u>Higher rk Catalan #s.</u>

<u>Thm</u> (Calaque - Enriquez - Etingof).

Let $V_n(\mu)$ = irrep of Gl_n assoc w/ dom wt μ.

and assume $\mu_1 + \cdots + \mu_n = -a$. Then

$$\boxed{\dim \mathrm{Hom}_{Gl_n}(V_n(\mu), \underset{M_{a/n}}{\underset{\uparrow}{\cdot}}) = \frac{1}{n} \dim V_n(\mu)}$$

$$\mathbb{C}[\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r)] = \bigoplus_{\substack{\nu \\ \text{partition with} \\ \leq \min(n,r) \text{ rows}}} V_n(\nu) \otimes V_r(\nu)^*$$

"So what's the dimension?"
"Zhenya! You have Russian version of Feynman syndrome"

<u>So</u>, $L_{a/n} = \bigoplus_{\nu : |\nu| = a} V_r(\nu)^{\oplus \frac{1}{n}\dim V_n(\nu)}$

<u>Cor</u>. $\dim L_{a/n} = \frac{1}{n}\binom{nr+a-1}{a}$.

For $r=1$, it is $\frac{(n+a-1)!}{n!\,a!}$ ← "rational Catalan #"

"~~Stupid~~ Obvious" torus gives quantized numbers.

<u>6</u> <u>Questions</u>. 1. Combinatorial meaning of $\dim L_{a/n}$?

2. Basis in $L_{a/n}$ - eigenbasis for certain comm. subalg. of $A_2(n,r)$?

3. Where's the "~~dumb~~ hidden" 1-dim torus?

⇝ Higher-rk (q,t) - Catalan #s.

$GL_r \times \mathbb{C}^\times \times \underline{\underline{\mathbb{C}^\times}} \curvearrowright \mathcal{M}(n,r)$?

Way #1. Maybe use Hodge filt.

Way #2. Use elliptic Hall alg $\curvearrowright \bigoplus_{n\geq 0} K_0(\mathrm{Coh}^{GL_n \times \mathbb{C}^\times}(\mathcal{M}(n,r))$.