Characterization of queer supercrystals

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Crystals of type A_n

Abstract crystal of type A_n: Nonempty set B with maps

$$e_i, f_i : B \to B \sqcup \{0\} \quad (1 \le i \le n)$$
$$\mathrm{wt} : B \to \Lambda$$

Index set: $I = \{1, \ldots, n\}$.
Weight lattice: $\Lambda = \mathbb{Z}_{\geqslant 0}^{n+1}$.
Simple roots: $\alpha_i = \epsilon_i - \epsilon_{i+1}$, ϵ_i i-th standard basis vector of $\mathbb{Z}^{n+1}$.
String lengths:

$$\varphi_i(x) = \max\{k \ge 0 \mid f_i^k x \ne 0\}, \; \varepsilon_i(x) = \max\{k \ge 0 \mid e_i^k x \ne 0\}$$

We require:

A1. $f_i x = x'$ if and only if $x = e_i x'$
$\mathrm{wt}(x) = \mathrm{wt}(x') + \alpha_i$

Crystals: Tensor product

Definition

Given crystals B and C of type A_n, the tensor product $B \otimes C$ has the following data.

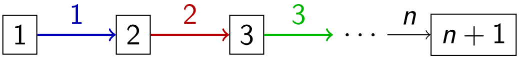
- Elements: $x \otimes y := (x, y) \in B \times C$
- Weight: $\mathrm{wt}(x \otimes y) = \mathrm{wt}(x) + \mathrm{wt}(y)$
- Crystal operators:

$$f_i(x \otimes y) = \begin{cases} f_i(x) \otimes y, & \text{if } \varphi_i(y) \leq \varepsilon_i(x), \\ x \otimes f_i(y), & \text{if } \varphi_i(y) > \varepsilon_i(x), \end{cases}$$

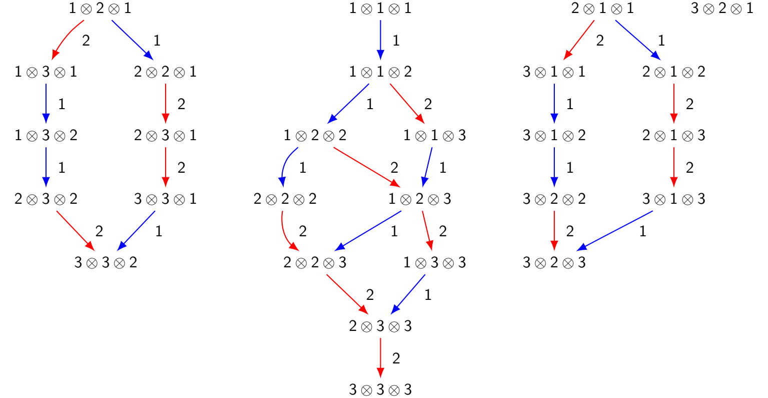
$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y, & \text{if } \varphi_i(y) < \varepsilon_i(x), \\ x \otimes e_i(y), & \text{if } \varphi_i(y) \geq \varepsilon_i(x). \end{cases}$$

Crystals: Examples

Standard crystal $\mathcal{B}$ of type A_n



Components of crystal of words $\mathcal{B}^{\otimes 3} = \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{B}$ of type A_2:



Lie superalgebras

- Superalgebra: $\mathbb{Z}_2$-graded algebra $A_0 \oplus A_1$, $A_i A_j \subseteq A_{i+j}$ for $i, j \in \mathbb{Z}_2$.
- Parity of $a \in A_i$ is $|a| = i$.
- Lie superalgebra: $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with
  - (Skew-supersymmetry) $[a, b] = -(-1)^{|a||b|}[b, a]$.
  - (Super Jacobi identity) $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$.

  for all $a \in \mathfrak{g}_i$, $b \in \mathfrak{g}_j$, $c \in \mathfrak{g}_k$.
- Queer Lie superalgebra, $\mathfrak{q}(n)$ is a super analogue of Lie algebra $\mathfrak{gl}(n)$.

$$\mathfrak{q}(n) = \left\{ \begin{bmatrix} A & B \\ B & A \end{bmatrix} \middle| \operatorname{tr}(B) = 0 \right\} \Big/ \mathbb{C} I_{2n}.$$

Queer supercrystals

- [Grantcharov, Jung, Kang, Kashiwara, Kim '10, '14]:
  Crystal bases of tensor representations of $\mathfrak{q}(n)$ using $U_q(\mathfrak{q}(n))$.
- Introduced queer supercrystals on words with tensor product rule.
- Explicit combinatorial realization of queer supercrystals using semistandard decomposition tableaux.
- Existence of fake highest (and lowest) weights on queer supercrystals.

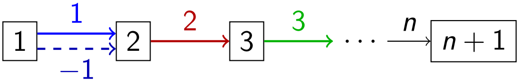
Why crystals?

Type A_n crystals	Type $\mathfrak{q}(n)$ crystals
Tensor representations of $\mathfrak{gl}(n)$	Tensor representations of $\mathfrak{q}(n)$
Irreps indexed by partitions λ	Irreps indexed by strict partitions λ
Crystals: $B(\lambda)$	Crystals: $\mathcal{B}(\lambda)$
$B(\lambda) \otimes B(\mu) \cong \bigoplus_\nu c^\nu_{\lambda\mu} B(\nu)$	$\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu) \cong \bigoplus_\nu g^\nu_{\lambda\mu} \mathcal{B}(\nu)$
Characters: Schur function, s_λ	Characters: Schur-P function, P_λ
$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$	$P_\lambda P_\mu = \sum_\nu g^\nu_{\lambda\mu} P_\nu$

- $c^\nu_{\lambda\mu}$ counts highest weight elements in $B(\lambda) \otimes B(\mu)$.
- $g^\nu_{\lambda\mu}$ counts highest weight elements in $\mathcal{B}(\lambda) \otimes \mathcal{B}(\mu)$.

Standard $\mathfrak{q}(n+1)$ crystal and tensor product

Standard crystal $\mathcal{B}$ of type $\mathfrak{q}(n+1)$:

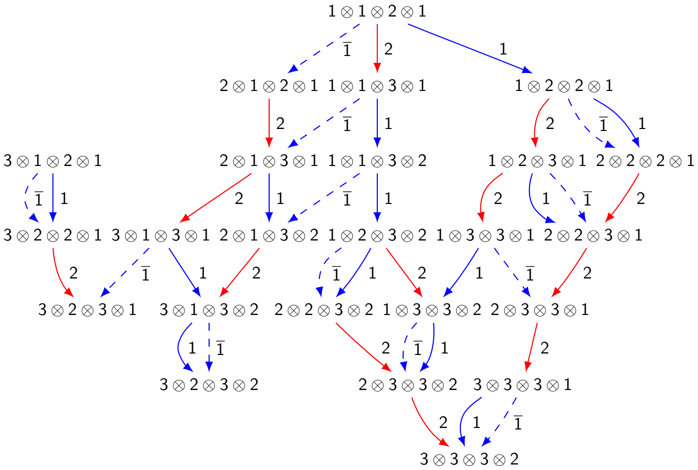


$$f_{-1}(x \otimes y) = \begin{cases} x \otimes f_{-1}(y), & \text{if } \mathrm{wt}(x)_1 = \mathrm{wt}(x)_2 = 0 \\ f_{-1}(x) \otimes y, & \text{otherwise,} \end{cases}$$

$$e_{-1}(x \otimes y) = \begin{cases} x \otimes e_{-1}(y), & \text{if } \mathrm{wt}(x)_1 = \mathrm{wt}(x)_2 = 0 \\ e_{-1}(x) \otimes y, & \text{otherwise.} \end{cases}$$

$\mathfrak{q}(n+1)$ crystal on Words

One connected component of $\mathcal{B}^{\otimes 4}$ for $\mathfrak{q}(3)$:



Queer supercrystals

How does one detect highest weight elements in queer supercrystals?

Definition

$$f_{-i} := s_{w_i^{-1}} f_{-1} s_{w_i}, \quad e_{-i} := s_{w_i^{-1}} e_{-1} s_{w_i},$$

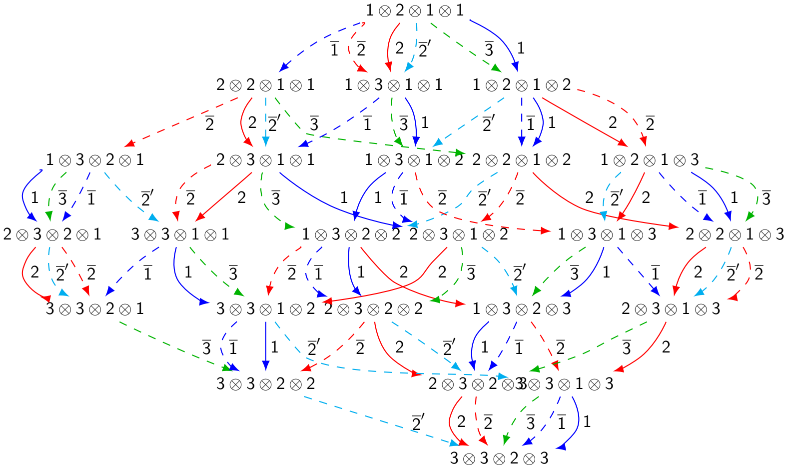
where $s_{w_i} = s_2 s_3 \dots s_i s_1 s_2 \dots s_{i-1}$ and s_i is the reflection along the i-th string.

Theorem (Grantcharov et al. '14)

Every connected component in $\mathcal{B}^{\otimes l}$ has a unique highest weight element u with $e_i u = 0$, $e_{-i} u = 0$ for all $i \in \{1, 2, \dots, n\}$.

Queer crystal: Example revisited

Same connected component of $\mathcal{B}^{\otimes 4}$:



Stembridge axioms

Question

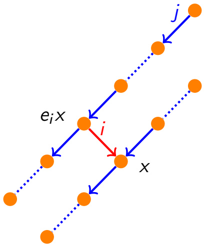
Is there a local characterization for a crystal graph?

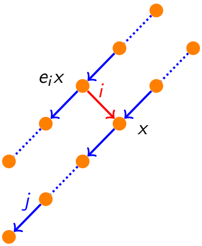
- [Stembridge '03] Yes, for crystals of simply-laced root systems (type A_n for example)
- Local rules characterizes Stembridge crystals, allows pure combinatorial analysis of crystals.

Stembridge axioms

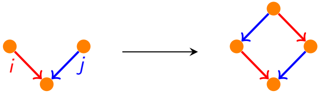
B is a crystal for a simply-laced root system with index set $I = \{1, 2 \ldots, n\}$.

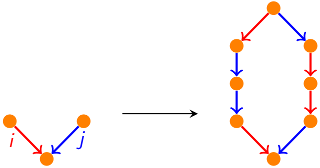
Axiom **S1.**





Stembridge axioms

Axiom **S2.** 

Axiom **S3.** 

Dual axioms **S2'.** and **S3'.** similarly hold.

Stembridge crystals: characterization

Why are Stembridge crystals important?

Theorem (Stembridge, '03)

- *Crystals* B, C *are Stembridge* $\Rightarrow B \otimes C$ *is Stembridge.*
- *Every connected component of a Stembridge crystal* B *has a unique highest weight element.*
- B, B' *are connected Stembridge crystals with* $u \in B$, $u' \in B'$ *as highest weight elements. If* $wt(u) = wt(u')$, *then* $B \cong B'$.

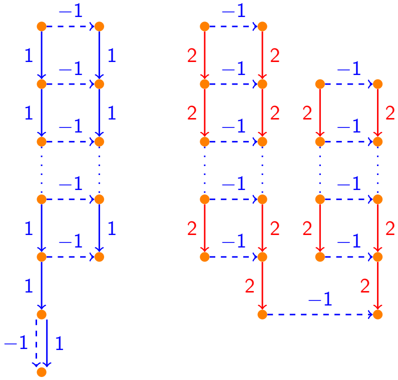
Question

Does there exist a similar characterization for queer supercrystals?

Local queer axioms

Conjecture (Assaf, Oguz 2018)

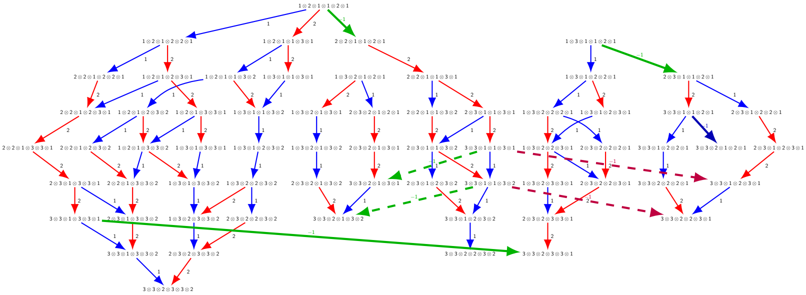
In addition to the Stembridge axioms, the following relations characterize type $\mathfrak{q}(n+1)$ crystals.



Counterexample

Example (Gillespie, Hawkes, P., Schilling 2018)

Consider $u = 1 \otimes 2 \otimes 1 \otimes 1 \otimes 2 \otimes 1 \in \mathcal{B}^{\otimes 6}$, $\mathrm{wt}(u) = (4, 2, 0)$:



Graph on type A_n components

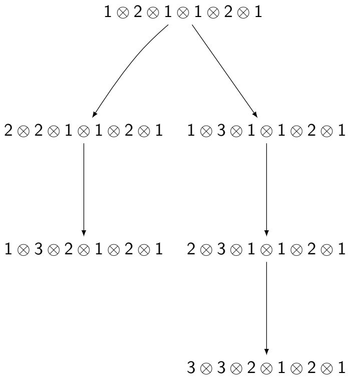
Set $I_0 = \{1, 2, \ldots, n\}$.
Let $\mathcal{C}$ be crystal with index set $I = I_0 \cup \{-1\}$ that is type A_n Stembridge when restricted to I_0.

Definition

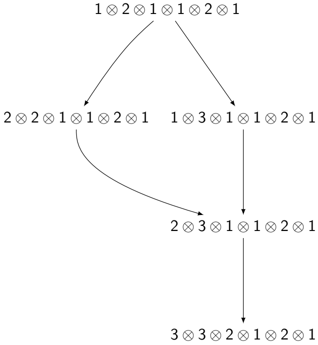
The component graph $G(\mathcal{C})$ has
Vertices: Type A_n components of $\mathcal{C}$, labeled by highest weight elements.
Edges: Component C_1 to C_2 has an edge if there are $x \in C_1$, $y \in C_2$ with $f_{-1}x = y$.

Graph on type A_n components: Counterexample

correct graph

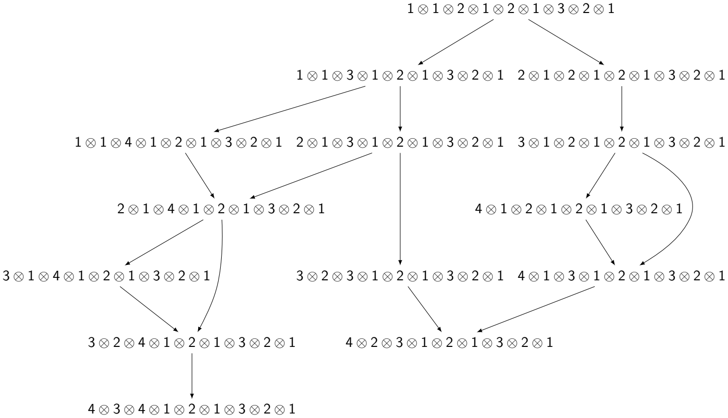


counterexample



Another example

- Consider $u = 1 \otimes 1 \otimes 2 \otimes 1 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 1 \in \mathcal{B}^{\otimes 9}$, $\mathrm{wt}(u) = (5, 3, 1, 0)$
- $P_{531} = s_{531} + s_{522} + s_{441} + s_{333} + 2s_{432} + s_{5211} + 2s_{4311} + 2s_{4221} + 2s_{3321} + s_{3222}$.



Connectivity Axioms

- Let C_1 be a type A_n component of queer crystal $\mathcal{C}$ and $x \in C_1$. Exactly one of the following three occurs:
  1. $f_{-1}(x) = 0$.
  2. $f_{-1}(x) \in C_1$.
  3. $f_{-1}(x) \in C_2$ for some type A_n component $C_2 \neq C_1$.
- If $f_{-1}(x) \neq 0$, then $f_{-1}(x)$ is expressible using type A_n operators on lowest weight elements of C_1 (resp. C_2).

Main theorem

Theorem (GHPS 2018)

Suppose that $\mathcal{C}$ is a connected abstract $\mathfrak{q}(n+1)$ crystals satisfying:

1. *$\mathcal{C}$ satisfies local queer axioms.*
2. *$\mathcal{C}$ satisfies the connectivity axioms* **C1. - C3.**
3. *$G(\mathcal{C}) \cong G(\mathcal{D})$, where $\mathcal{D}$ is a connected component of $\mathcal{B}^{\otimes l}$.*

Then as queer supercrystals, $\mathcal{C} \cong \mathcal{D}$.

Thank you!