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White-Noise and Geometrical Optics Limits of Wigner–Moyal Equation for Beam Waves in Turbulent Media II: Two-Frequency Formulation

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We introduce two-frequency Wigner distribution in the setting of parabolic approximation to study the scaling limits of the wave propagation in a turbulent medium at two different frequencies. We show that the two-frequency Wigner distribution satisfies a closed-form equation (the two-frequency Wigner–Moyal equation). In the white-noise limit we show the convergence of weak solutions of the two-frequency Wigner–Moyal equation to a Markovian model and thus prove rigorously the Markovian approximation with power-spectral densities widely used in the physics literature. We also prove the convergence of the simultaneous geometrical optics limit whose mean field equation has a simple, universal form and is exactly solvable.

KEY WORDS: Two-frequency Wigner distribution; martingale; geometrical optics; turbulent media.

1. INTRODUCTION

High-data-rate communication systems at millimeter and optical frequencies, remote sensing and detection and the astronomical imaging all require understanding of stochastic pulse propagation. Whether the background random medium is dispersive (such as electromagnetic waves in the ionosphere, interplanetary and interstellar media) or non-dispersive (such as electromagnetic waves in the atmosphere) analysis of pulsed signal propagation is usually based on spectral decomposition of the time-dependent signal into time-harmonic wave fields.

In this formulation, the complete information about transient propagation requires a solution for the statistical moments of the wave field at

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different frequencies and locations. In particular the two-frequency mutual coherence provides a measure of the coherence bandwidth. In the context of optical wave propagating through the atmosphere, analysis accounting for multiple scattering and long propagation distances are based on the paraxial equation for the complex amplitude and the Markov approximation.(9,14) In this framework, a complete set of two-frequency moment equations can be derived.

The purposes of this work are first to establish a general two-frequency framework (without Markov approximation) in terms of the two-frequency Wigner distribution, secondly, to use this framework to prove rigorously the Markov approximation in the white-noise scaling limit and thirdly to obtain the geometrical optics limit of the white-noise model. One of the main features (Theorems 1 and 2, Section 3) of our approach is the closed form Eq. (42) for the mean two-frequency Wigner distribution in the geometrical optics limit, which takes a universal form and is exactly solvable, (see Appendix B).

All of these will be carried out for the random turbulent medium with a power-law spectrum. This has been accomplished for the one-frequency setting in ref. 5; the present work is the generalization of the previous approach to the two-frequency setting. Although here we will be treating, as an example, the case of optical wave propagating through the turbulent atmosphere, our approach is entirely suitable for dispersive and/or dissipative media after proper modification of the refractive index field to include frequency dependence and/or lossiness.

In the paraxial approximation (9,14) for the time-harmonic component the wave amplitude Ψ_j, $j=1,2$, at two different wavenumbers k_j are given by the solutions of the parabolic wave equation, which after non-dimensionalization with respect to some reference lengths L_z and L_x in the longitudinal and transverse directions, respectively, has this form

$$i\frac{\partial\Psi_j}{\partial z}+\frac{\gamma}{2\tilde{k}_j}\Delta\Psi_j+\tilde{k}_j k_0 L_z\tilde{n}(zL_z,\mathbf{x}L_x)\Psi_j=0,\quad j=1,2,\tag{1}$$

where $\tilde{k}_j=k_j/k_0$, $j=1,2$ are the normalized wavenumbers with respect to the central wavenumber k_0 and γ is the Fresnel number

$$\gamma=\frac{L_z}{k_0L_x^2}.\tag{2}$$

Here $\tilde{n}(\vec{\mathbf{x}})$, $\vec{\mathbf{x}}=(z,\mathbf{x})\in\mathbb{R}^{d+1}$ is the relative fluctuation of the refractive index field, which is a homogeneous, square-integrable random field and, as in ref. 5 is assumed to have a spectral density $\Phi(\vec{\mathbf{k}})$ satisfying the upper bound

$$\Phi(\vec{\mathbf{k}}) \leqslant K(L_0^{-2}+|\vec{\mathbf{k}}|^2)^{-H-1/2-d/2}\left(1+\ell_0^{-2}|\mathbf{k}|^2\right)^{-2}$$
$$\vec{\mathbf{k}}=(\xi,\mathbf{k})\in\mathbb{R}^{d+1}, \quad H\in(0,1) \tag{3}$$

for some positive constant $K<\infty$ where L_0 and ℓ_0 in (3) are, respectively, the outer and inner scales of the turbulent medium and H is the Hurst exponent of the random field. A relevant example is the generalized von Kármán spectral density[12] with $H=1/3$.

In terms of the non-dimensional parameters

$$\varepsilon=\sqrt{\frac{L_x}{L_z}}, \qquad \eta=\frac{L_x}{L_0}, \quad \rho=\frac{L_x}{\ell_0},$$

we rewrite (1) as

$$i\frac{\partial\Psi_j^\varepsilon}{\partial z}+\frac{\gamma}{2\tilde{k}_j}\Delta\Psi_j^\varepsilon+\frac{\tilde{k}_j}{\gamma}\frac{\mu}{\varepsilon}V\left(\frac{z}{\varepsilon^2},\mathbf{x}\right)\Psi_j^\varepsilon=0, \quad \Psi_j^\varepsilon(0,\mathbf{x})=\Psi_{j,0}(\mathbf{x}), \quad j=1,2 \tag{4}$$

with

$$\mu=\frac{\sigma L_x^H}{\varepsilon^3}, \tag{5}$$

where σ the standard variation of the homogeneous field $\tilde{n}(z,\mathbf{x})$ and V is the normalized refractive index field with a spectral density satisfying the upper bound

$$\Phi_{\eta,\rho}(\vec{\mathbf{k}}) \leqslant K(\eta^2+|\vec{\mathbf{k}}|^2)^{-H-1/2-d/2}\left(1+\rho^{-2}|\vec{\mathbf{k}}|^2\right)^{-2},$$
$$\vec{\mathbf{k}}\in\mathbb{R}^{d+1}, \quad H\in(0,1) \tag{6}$$

for some positive constant K.

The white-noise scaling corresponds to $\varepsilon\to 0$ with a fixed μ. For convenience we set $\mu=1$. If the observation scales L_z and L_x are the longitudinal and transverse scales, respectively, of the wave beam then $\varepsilon\ll 1$ corresponds to a long, narrow wave beam. The white-noise scaling limit of Eq. (4) is analyzed in refs. 3 and 6. The limit $\gamma\to 0$ corresponds to the geometrical optics limit. The parabolic approximation has been widely used in the literature on waves in turbulent media (see for example refs. 9 and 14 and the references therein). But to our knowledge the parabolic approximation of the reduced wave equation has not been proved in the present context. However, see ref. 2 for an interesting result of a simultaneous limit of parabolic and white-noise scaling for a layered medium with $\tilde{n}=\tilde{n}(z)$.

Although we do not assume isotropic spectral densities, the spectral density always satisfies the basic symmetry:

$$\Phi_{(\eta,\rho)}(\xi,\mathbf{k})=\Phi_{(\eta,\rho)}(-\xi,\mathbf{k})=\Phi_{(\eta,\rho)}(\xi,-\mathbf{k}),\quad \forall(\xi,\mathbf{k})\in\mathbb{R}^{d+1} \tag{7}$$

because the refractive-index field is real-valued. We also assume that $V_z(\mathbf{x})\equiv V(z,\mathbf{x})$ is a centered, square-integrable, z-stationary and $\mathbf{x}$-homogeneous process with the (partial) spectral representation

$$V_z(\mathbf{x})=\int \exp(i\mathbf{p}\cdot\mathbf{x})\widehat{V}_z(d\mathbf{p}), \tag{8}$$

where the process $\widehat{V}_z(d\mathbf{p})$ is the z-stationary orthogonal spectral measure satisfying

$$\mathbb{E}\left[\hat{V}_z(d\mathbf{p})\hat{V}_z(d\mathbf{q})\right]=\delta(\mathbf{p}+\mathbf{q})\left[\int \Phi(w,\mathbf{p})dw\right]d\mathbf{p}\,d\mathbf{q}. \tag{9}$$

We do *not* assume the Gaussian property but instead a sub-Gaussian property (see Section 3.2 for precise statements).

1.1. Wigner Distribution and Wigner–Moyal Equation

We introduce two-frequency Wigner distributions

$$W_z^\varepsilon(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi_1^\varepsilon\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_2^{\varepsilon*}\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)d\mathbf{y} \tag{10}$$

and its complex conjugate $W^{\varepsilon*}$ which are ideally suited for analyzing the two-frequency problem.

The following bounds can be derived easily from (10)

$$\|W_z^\varepsilon\|_\infty\leqslant\left(\frac{\sqrt{\tilde{k}_1\tilde{k}_2}}{2\gamma\pi}\right)^d\|\Psi_1^\varepsilon(z,\cdot)\|_2\|\Psi_2^\varepsilon(z,\cdot)\|_2,$$

$$\|W_z^\varepsilon\|_2=\left(\frac{\sqrt{\tilde{k}_1\tilde{k}_2}}{2\gamma\pi}\right)^{d/2}\|\Psi_1^\varepsilon(z,\cdot)\|_2\|\Psi_2^\varepsilon(z,\cdot)\|_2.$$

Hence

$$\|W_z^\varepsilon\|_\infty \leqslant \left(\frac{\sqrt{\tilde{k}_1\tilde{k}_2}}{2\gamma\pi}\right)^d \|\Psi_1(0,\cdot)\|_2\|\Psi_2(0,\cdot)\|_2, \tag{11}$$

$$\|W_z^\varepsilon\|_2 = \|W_0\|_2 \tag{12}$$

The Wigner distribution has the following obvious properties.

$$\int W_z^\varepsilon(\mathbf{x},\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{y}}d\mathbf{p} = \Psi_1^\varepsilon\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_2^{\varepsilon*}\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right) \tag{13}$$

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} W_z^\varepsilon(\mathbf{x},\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{q}}d\mathbf{x} = \left(\frac{\sqrt{\tilde{k}_1\tilde{k}_2}}{\gamma}\right)^d \widehat{\Psi}_1^\varepsilon\left(z,\frac{\mathbf{p}\sqrt{\tilde{k}_2}}{\gamma}+\frac{\sqrt{\tilde{k}_1}\mathbf{q}}{2}\right)\widehat{\Psi}_2^{\varepsilon*}\left(z,\frac{\mathbf{p}\sqrt{\tilde{k}_2}}{\gamma}-\frac{\sqrt{\tilde{k}_1}\mathbf{q}}{2}\right). \tag{14}$$

The Wigner distribution W_z^ε satisfies the Wigner–Moyal equation

$$\frac{\partial W_z^\varepsilon}{\partial z}+\mathbf{p}\cdot\nabla_\mathbf{x} W_z^\varepsilon+\frac{1}{\varepsilon}\mathcal{L}_z^\varepsilon W_z^\varepsilon=0 \tag{15}$$

with the initial data

$$W_0(\mathbf{x},\mathbf{k})=\frac{1}{(2\pi)^d}\int e^{i\mathbf{k}\cdot\mathbf{y}}\Psi_{1,0}\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_{2,0}^*\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)d\mathbf{y}, \tag{16}$$

where the operator $\mathcal{L}_z^\varepsilon$ is formally given as

$$\mathcal{L}_z^\varepsilon W_z^\varepsilon = i\int \gamma^{-1}\left[e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{\tilde{k}_1}}\tilde{k}_1 W_z^\varepsilon\left(\mathbf{x},\mathbf{p}+\frac{\gamma\mathbf{q}}{2\sqrt{\tilde{k}_1}}\right)\right.$$
$$\left. - e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{\tilde{k}_2}}\tilde{k}_2 W_z^\varepsilon\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2\sqrt{\tilde{k}_2}}\right)\right]\widehat{V}\left(\frac{z}{\varepsilon^2},d\mathbf{q}\right).$$

Equation (15) can be formally derived as follows. Differentiating (10) w.r.t. z and using (4) we have

$$\begin{aligned}
\frac{\partial W_z^\varepsilon}{\partial z}(\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi)^d}\int e^{-i\mathbf{y}\cdot\mathbf{p}}\left[\frac{i\gamma}{2\tilde{k}_1}\Delta\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\right.\\
&\quad\left.-\frac{i\gamma}{2\tilde{k}_2}\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}}}\right)\Delta\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\right]d\mathbf{y}\\
&\quad+\frac{1}{(2\pi)^d}\int e^{-i\mathbf{y}\cdot\mathbf{p}}\left[\frac{i\tilde{k}_1}{\gamma}V\left(\frac{z}{\varepsilon^2},\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\right.\\
&\quad\times\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)-\frac{i\tilde{k}_2}{\gamma}V\left(\frac{z}{\varepsilon^2},\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\\
&\quad\left.\times\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\right]d\mathbf{y},
\end{aligned}$$

which can be written as

$$\begin{aligned}
&\frac{\partial W_z^\varepsilon}{\partial z}(\mathbf{x},\mathbf{p})\\
&=\frac{1}{(2\pi)^d}\int e^{-i\mathbf{y}\cdot\mathbf{p}}\left[\frac{i}{\sqrt{\tilde{k}_1}}\nabla_{\mathbf{y}}\cdot\left[\nabla\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\right]\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\right.\\
&\quad\left.+\frac{i}{\sqrt{\tilde{k}_2}}\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\nabla_{\mathbf{y}}\cdot\left[\nabla\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\right]\right]d\mathbf{y}\\
&\quad+\frac{1}{(2\pi)^d}\int e^{-i\mathbf{y}\cdot\mathbf{p}}\left[\frac{i\tilde{k}_1}{\gamma}\int d\hat{V}\left(\frac{z}{\varepsilon^2},\mathbf{q}\right)e^{i\mathbf{q}\cdot(\mathbf{x}+\gamma\mathbf{y}/2)\tilde{k}_1^{-1/2}}\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\right.\\
&\quad\times\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)-\frac{i\tilde{k}_2}{\gamma}\int d\hat{V}\left(\frac{z}{\varepsilon^2},\mathbf{q}\right)e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y}/2)\tilde{k}_2^{-1/2}}\\
&\quad\left.\times\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)\right]d\mathbf{y}
\end{aligned}$$

by using the spectral representation (8). Integrating by parts and expressing the right side in terms of W_z^ε we obtain Eq. (15). Note the cancellation of the term

$$\frac{1}{(2\pi)^d}\int e^{-i\mathbf{y}\cdot\mathbf{p}}\frac{i\gamma}{2\sqrt{\tilde{k}_1\tilde{k}_2}}\nabla\Psi_1\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right)\cdot\nabla\Psi_2^*\left(z,\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)d\mathbf{y}$$

in the process of integrating by parts.

The complex conjugate $W_z^{\varepsilon*}(\mathbf{x},\mathbf{p})$ satisfies a similar equation

$$\frac{\partial W_z^{\varepsilon*}}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}W_z^{\varepsilon*}+\frac{1}{\varepsilon}\mathcal{L}_z^{\varepsilon*}W_z^{\varepsilon*}=0, \tag{17}$$

where

$$\begin{aligned}\mathcal{L}_z^{\varepsilon*}W_z^{\varepsilon*}=i\int\gamma^{-1}\Bigg[&e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{\tilde{k}_2}}\tilde{k}_2W_z^{\varepsilon*}\left(\mathbf{x},\mathbf{p}+\frac{\gamma\mathbf{q}}{2\sqrt{\tilde{k}_2}}\right)\\&-e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{\tilde{k}_1}}\tilde{k}_1W_z^{\varepsilon*}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2\sqrt{\tilde{k}_1}}\right)\Bigg]\widehat{V}\left(\frac{z}{\varepsilon^2},d\mathbf{q}\right).\end{aligned}$$

We use the following definition of the Fourier transform and inversion:

$$\mathcal{F}f(\mathbf{p})=\frac{1}{(2\pi)^d}\int e^{-i\mathbf{x}\cdot\mathbf{p}}f(\mathbf{x})d\mathbf{x}$$
$$\mathcal{F}^{-1}g(\mathbf{x})=\int e^{i\mathbf{p}\cdot\mathbf{x}}g(\mathbf{p})d\mathbf{p},$$

when making a *partial* (inverse) Fourier transform on a phase-space function we will write $\mathcal{F}_1$ (resp. $\mathcal{F}_1^{-1}$) and $\mathcal{F}_2$ (resp. $\mathcal{F}_2^{-1}$) to denote the (resp. inverse) transform w.r.t. $\mathbf{x}$ and $\mathbf{p}$, respectively.

In this paper we consider the weak formulation of the Wigner–Moyal equation: To find $W_z^\varepsilon\in C([0,\infty);L^2_w(\mathbb{R}^{2d}))$ such that $\|W_z^\varepsilon\|_2\leqslant\|W_0\|_2,\forall z>0$, (cf. Eq. (12) and Section 1.2) and

$$\left\langle W_z^\varepsilon,\theta\right\rangle-\langle W_0,\theta\rangle=\int_0^z\left\langle W_s^\varepsilon,\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle ds+\frac{1}{\varepsilon}\int_0^z\left\langle W_s^\varepsilon,\mathcal{L}_s^{\varepsilon*}\theta\right\rangle ds,\quad\forall\theta\in\mathcal{S}, \tag{18}$$

where

$$\mathcal{S}=\left\{\theta(\mathbf{x},\mathbf{p})\in L^2(\mathbb{R}^{2d});\,\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\in C_c^\infty(\mathbb{R}^{2d})\right\}.$$

Here and below $L^2_w(\mathbb{R}^{2d})$ is the space of complex-valued square integrable functions on the phase space $\mathbb{R}^{2d}$ endowed with the weak topology and the inner product

$$\langle W_1, W_2\rangle=\int W_1^*(\mathbf{x},\mathbf{p})W_2(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p}.$$

We define for every realization of V_z^ε the operator $\mathcal{L}_z^{\varepsilon *}$ to act on a phase-space test function θ as

$$\mathcal{L}_z^{\varepsilon *}\theta(\mathbf{x},\mathbf{p})\equiv -i\gamma^{-1}\mathcal{F}_2\left[\delta_{21}V_z^\varepsilon(\mathbf{x},\mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right] \tag{19}$$

with the difference operator δ_{21} given by

$$\delta_{21}V_z^\varepsilon(\mathbf{x},\mathbf{y}) \equiv \tilde{k}_1 V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}}+\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right)-\tilde{k}_2 V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}}-\frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right) \tag{20}$$

$$V_z^\varepsilon(\mathbf{x}) = V_{z/\varepsilon^2}(\mathbf{x}). \tag{21}$$

We define $\mathcal{L}_z^\varepsilon$ in the same way.

A main advantage of the formulation with the Wigner distribution is the possibility of obtaining a closed form Eq. (42) in the geometrical optics limit. Another advantage is its capability of dealing with the mixed-state initial data, which are the convex combinations of the pure-state Wigner distribution (10).

1.2. Existence of Weak Solutions

The existence of weak solutions can be established by the weak compactness argument as follows. Without loss of generality we set $\varepsilon=1$.

First, we introduce truncation $N<\infty$

$$V^{(N)}(z,\mathbf{x})=\mathbb{I}_N V(z,\mathbf{x}),$$

where $\mathbb{I}_N$ is the characteristic function of the set $\{|V(z,x)|<N\}$. For such bounded V_N the corresponding operator as given by

$$\mathcal{L}_z^{(N)} W \otimes \mathcal{L}_z^{(N)*} W^* \equiv i\gamma^{-1}\mathcal{F}_2\left(\left[\delta_{12}V_z^{(N)}\mathcal{F}_2^{-1}\right]W \otimes \left[\delta_{21}V_z^{(N)}\mathcal{F}_2^{-1}W^*\right]\right)$$

is a bounded skew-adjoint operator on $L^2(\mathbb{R}^{2d})\otimes L^2(\mathbb{R}^{2d})$. Hence the corresponding system of Wigner–Moyal equations gives rise to a C_0-group of unitary maps on $L^2\otimes L^2$. Let us denote the solution by $(W_z^{(N)}, W_z^{(N)*})$. Passing to the limit $N\to\infty$ by selecting a weakly convergent subsequence we obtain a L^2-weak solution for the Wigner–Moyal equation with the truncation removed if V is locally square-integrable as is assumed here. The limiting solution W_z has a L^2-norm equal to or less than that of W_0.

Moreover, from Eq. (18), it is easy to see that $\left\langle W_z^{(N)},\theta\right\rangle$ is equi-continuous on any compact subset of $z\in\mathbb{R}$. By Arzela–Ascoli lemma, $\langle W_z,\theta\rangle$ is z-continuous almost surely. Because $\left\langle W_z^{(N)},\theta\right\rangle$ is adapted to the filtration of V_z and the convergence is almost sure, the resulting solution W_z is adapted to the filtration of V_z.

We will not address the uniqueness of solution for the Wigner–Moyal Eq. (18) but we will show that as $\varepsilon\to 0$ any sequence of weak solutions to Eq. (18) converges in a suitable sense to the unique solution of a martingale problem (see Theorems 1 and 2).

1.3. Geometrical Optics Limit

Consider the simultaneous limit

$$\gamma\to 0,\quad \tilde{k}_1,\tilde{k}_2\to\tilde{k}\neq 0 \tag{22}$$

such that

$$\frac{\tilde{k}_2-\tilde{k}_1}{\gamma}=\frac{(k_2-k_1)L_x^2}{L_z}\to\beta\geqslant 0. \tag{23}$$

Here the parameter $\beta<\infty$ has the physical meaning of a normalized bandwidth. We still assume $W_0(\mathbf{x},\mathbf{p})\in L^2(\mathbb{R}^{2d})$ independent of γ.

When acting on the test function space $\mathcal{S}$, $\mathcal{L}_z^\varepsilon$ has the following limit

$$\begin{aligned}\lim_{\gamma\to 0}\mathcal{L}_z^{\varepsilon *}\theta(\mathbf{x},\mathbf{p}) &= -\mathcal{F}_2\left[\nabla_{\mathbf{x}}V_z^\varepsilon(\mathbf{x})\cdot\left[i\mathbf{y}\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right]\right]\\ &= -\sqrt{\tilde{k}}\nabla V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}}}\right)\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})+i\beta\theta(\mathbf{x},\mathbf{p})\\ &\quad\times\left[V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}}}\right)-\frac{\mathbf{x}}{2\sqrt{\tilde{k}}}\cdot\nabla V_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}}}\right)\right]\end{aligned}\tag{24}$$

in the L^2-sense for all $\theta\in\mathcal{S}$ and all locally square-integrable V_z^ε.

2. THE WHITE-NOISE MODELS

Now we formulate the solutions for the white-noise model as the solutions to the corresponding martingale problem: Find the law $\mathbb{Q}$ on $\mathcal{Z}=C([0,\infty);L_w^2(\mathbb{R}^{2d}))$ such that for $\zeta\in\mathcal{Z}$ and $W_z(\omega)\equiv\zeta(z), z\geqslant 0$ we have that $\mathbb{Q}\left(W_0(\omega)=W_0\in L^2(\mathbb{R}^{2d})\right)=1$ and that

$$\begin{aligned}f(\langle W_z,\theta\rangle)-\int_0^z\Big\{&f'(\langle W_s,\theta\rangle)\left[\langle W_s,\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle+\left\langle W_s,\overline{\mathcal{Q}}_0\theta\right\rangle\right]\\ &+f''(\langle W_s,\theta\rangle)\left\langle W_s,\overline{\mathcal{K}}_\theta W_s\right\rangle\Big\}\,ds\\ &\text{with}\quad \overline{\mathcal{K}}_\theta W_s=\int\overline{\mathcal{Q}}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})W_s(\mathbf{y},\mathbf{q})\,d\mathbf{y}\,d\mathbf{q}\end{aligned}$$

is a martingale for each $f\in C^\infty(\mathbb{R})$.

Here, in the case of the white-noise model for the Wigner–Moyal equation (Theorem 1), the covariance operators $\overline{\mathcal{Q}}$, $\overline{\mathcal{Q}}_0$ are defined as

$$\begin{aligned}&\overline{\mathcal{Q}}_0\theta(\mathbf{x},\mathbf{p})\\ &=\int\Phi_\eta^\infty(\mathbf{q})\gamma^{-2}\Big[\tilde{k}_1\tilde{k}_2e^{i(\tilde{k}_1^{-1/2}-\tilde{k}_2^{-1/2})\mathbf{q}\cdot\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-(\tilde{k}_1^{-1/2}+\tilde{k}_2^{-1/2})\gamma\mathbf{q}/2)\\ &\quad+\tilde{k}_1\tilde{k}_2e^{-i(\tilde{k}_1^{-1/2}-\tilde{k}_2^{-1/2})\mathbf{q}\cdot\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+(\tilde{k}_1^{-1/2}+\tilde{k}_2^{-1/2})\gamma\mathbf{q}/2)-(\tilde{k}_1^2+\tilde{k}_2^2)\theta(\mathbf{x},\mathbf{p})\Big]d\mathbf{q}.\end{aligned}\tag{25}$$

$$
\begin{aligned}
&\overline{\mathcal{Q}}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&=\int \Phi^{\infty}_{\eta}(\mathbf{p}')\gamma^{-2}\Big[e^{i\mathbf{p}'\cdot\mathbf{x}\tilde{k}_1^{-1/2}}\tilde{k}_1\theta(\mathbf{x},\mathbf{p}-\tilde{k}_1^{-1/2}\gamma\mathbf{p}'/2)\\
&\qquad\qquad -e^{i\mathbf{p}'\cdot\mathbf{x}\tilde{k}_2^{-1/2}}\tilde{k}_2\theta(\mathbf{x},\mathbf{p}+\tilde{k}_2^{-1/2}\gamma\mathbf{p}'/2)\Big]\\
&\quad\times\Big[e^{-i\mathbf{p}'\cdot\mathbf{y}\tilde{k}_2^{-1/2}}\tilde{k}_2\theta(\mathbf{y},\mathbf{q}-\tilde{k}_2^{-1/2}\gamma\mathbf{p}'/2)-e^{-i\mathbf{p}'\cdot\mathbf{y}\tilde{k}_1^{-1/2}}\tilde{k}_1\theta(\mathbf{y},\mathbf{q}+\tilde{k}_1^{-1/2}\gamma\mathbf{p}'/2)\Big]\,d\mathbf{p}'
\end{aligned}
\tag{26}
$$

and, in the case of the geometrical optics white-noise limit (Theorem 2),

$$
\overline{\mathcal{Q}}_0\theta = -\tilde{k}\nabla_{\mathbf{p}}\cdot\mathbf{D}(0)\cdot\nabla_{\mathbf{p}}\theta - i\beta\mathbf{x}\cdot\mathbf{D}(0)\cdot\nabla_{\mathbf{p}}\theta+\beta^2D_0(0)\theta+\frac{\beta^2}{4\tilde{k}}\mathbf{x}\cdot\mathbf{D}(0)\cdot\mathbf{x}\theta \tag{27}
$$

$$
\begin{aligned}
&\overline{\mathcal{Q}}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&=\int\Phi^{\rho}_{\eta}(\mathbf{q}')e^{i\mathbf{q}'\cdot(\mathbf{x}-\mathbf{y})\tilde{k}^{-1/2}}\Big[\tilde{k}^{1/2}\mathbf{q}'\cdot\nabla_{\mathbf{p}}+\beta-i2^{-1}\tilde{k}^{-1/2}\beta\mathbf{q}'\cdot\mathbf{x}\Big]\\
&\quad\otimes\Big[\tilde{k}^{1/2}\mathbf{q}'\cdot\nabla_{\mathbf{q}}-\beta-i2^{-1}\tilde{k}^{-1/2}\beta\mathbf{q}'\cdot\mathbf{y}\Big]\,d\mathbf{q}'\theta(\mathbf{x},\mathbf{p})\otimes\theta(\mathbf{y},\mathbf{q})\\
&=\tilde{k}\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\nabla_{\mathbf{q}}\theta(\mathbf{y},\mathbf{q})-\frac{\beta^2}{4\tilde{k}}\mathbf{x}\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\mathbf{y}\theta(\mathbf{x},\mathbf{p})\theta(\mathbf{y},\mathbf{q})\\
&\quad-\beta^2D_0(\mathbf{x}-\mathbf{y})\theta(\mathbf{x},\mathbf{p})\theta(\mathbf{y},\mathbf{q})+\tilde{k}^{1/2}\beta\mathbf{D}'(\mathbf{x}-\mathbf{y})\cdot(\nabla_{\mathbf{q}}-\nabla_{\mathbf{p}})\theta(\mathbf{x},\mathbf{p})\theta(\mathbf{y},\mathbf{q})\\
&\quad-i2^{-1}\beta\left[\mathbf{y}\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\nabla_{\mathbf{p}}+\mathbf{x}\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\nabla_{\mathbf{q}}\right]\theta(\mathbf{x},\mathbf{p})\theta(\mathbf{y},\mathbf{q})\\
&\quad+i2^{-1}\tilde{k}^{-1/2}\beta^2(\mathbf{x}-\mathbf{y})\cdot\mathbf{D}'(\mathbf{x}-\mathbf{y})\theta(\mathbf{x},\mathbf{p})\theta(\mathbf{y},\mathbf{q})
\end{aligned}
\tag{28}
$$

where

$$
\Phi^{\rho}_{\eta}(\mathbf{q})\equiv\Phi_{\eta,\rho}(0,\mathbf{q}),\quad \eta\geqslant 0,\quad \rho\leqslant\infty, \tag{29}
$$

$$
\mathbf{D}(\mathbf{x}-\mathbf{y})=\int e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})\tilde{k}^{-1/2}}\Phi^{\rho}_{\eta}(\mathbf{q})\mathbf{q}\otimes\mathbf{q}\,d\mathbf{q}, \tag{30}
$$

$$
\mathbf{D}'(\mathbf{x}-\mathbf{y})=\int e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})\tilde{k}^{-1/2}}\Phi^{\rho}_{\eta}(\mathbf{q})\mathbf{q}\,d\mathbf{q}, \tag{31}
$$

$$
D_0(\mathbf{x}-\mathbf{y})=\int e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})\tilde{k}^{-1/2}}\Phi^{\rho}_{\eta}(\mathbf{q})d\mathbf{q}. \tag{32}
$$

To obtain Eqs. (27) and (28) from (25) and (26), respectively, in the narrow-band geometrical optics limits (22) and (23) we write

$$
\tilde{k}_1=\tilde{k}-\beta\gamma/2,\qquad \tilde{k}_2=\tilde{k}+\beta\gamma/2.
$$

Using the approximation

$$\tilde{k}_1^{-1/2} - \tilde{k}_2^{-1/2} \approx \tilde{k}^{-1/2}\left[\frac{\beta\gamma}{2\tilde{k}} + \frac{7}{8}\left(\frac{\beta\gamma}{2\tilde{k}}\right)^3\right], \tag{33}$$

$$\tilde{k}_1^{-1/2} + \tilde{k}_2^{-1/2} \approx \tilde{k}^{-1/2}\left[2 + \frac{3}{4}\left(\frac{\beta\gamma}{2\tilde{k}}\right)^2\right] \tag{34}$$

in (25) and expand

$$\exp\left[\pm i(\tilde{k}_1^{-1/2} - \tilde{k}_2^{-1/2})\mathbf{q}\cdot\mathbf{x}\right] \quad \text{and} \quad W(\mathbf{x}, \mathbf{p} \pm (\tilde{k}_1^{-1/2} + \tilde{k}_2^{-1/2})\gamma\mathbf{q}/2)$$

up to the second order in γ.

In the worst case scenario allowed by the bound (6) (cf. (60)) the functions $\mathbf{D}, \mathbf{D}'$ and D_0 have the following near and far field behaviors

$$\begin{gathered} \mathbf{D}(\mathbf{x}-\mathbf{y}) = O(|\mathbf{x}-\mathbf{y}|^{2H-1}), \qquad \mathbf{D}'(\mathbf{x}-\mathbf{y}) = O(|\mathbf{x}-\mathbf{y}|^{2H}), \\ D_0(\mathbf{x}-\mathbf{y}) = O(|\mathbf{x}-\mathbf{y}|^{2H+1}) \end{gathered} \tag{35}$$

for $\eta = 0, |\mathbf{x}-\mathbf{y}| \gg 1$ or $\rho = \infty, |\mathbf{x}-\mathbf{y}| \ll 1$. Hence the operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}_0$ are well-defined for any test function $\theta \in \mathcal{S}$ in the former case for any $H \in (0, 1), \eta > 0$ or $\eta = 0, H \in (0, 1/2)$, and in the latter case for $H \in (0, 1), 0 < \eta < \rho < \infty$ or $H \in (0, 1/2), 0 = \eta < \rho < \infty$ or $H \in (1/2, 1), 0 < \eta < \rho = \infty$.

That the martingale problem as formulated with the special class of test functions is sufficient to characterize the law $\mathbb{Q}$ follows from the uniqueness result discussed in Section 2.2.

To see that (26) is square-integrable and well-defined for any $L^2(\mathbb{R}^{2d})$-valued process W_z, we apply $\mathcal{F}_2^{-1}$ to (26) and obtain

$$\begin{aligned} \mathcal{F}_2^{-1}\overline{\mathcal{K}}_\theta W_s(\mathbf{x}, \mathbf{x}') = {} & (2\pi)^{-d}\mathcal{F}_2^{-1}\theta(\mathbf{x}, \mathbf{x}') \int \Phi_\eta^\infty(\mathbf{p}')\mathcal{F}_2^{-1}\theta(\mathbf{y}, \mathbf{y}')\mathcal{F}_2^{-1}W_z(\mathbf{y}, -\mathbf{y}')\gamma^{-2} \\ & \times \left[\tilde{k}_2 e^{-i\mathbf{p}'\cdot(\mathbf{y}-\gamma\mathbf{y}'/2)\tilde{k}_2^{-1/2}} - \tilde{k}_1 e^{-i\mathbf{p}'\cdot(\mathbf{y}+\gamma\mathbf{y}'/2)\tilde{k}_1^{-1/2}}\right] \\ & \times \left[\tilde{k}_1 e^{i\mathbf{p}'\cdot(\mathbf{x}+\gamma\mathbf{x}'/2)\tilde{k}_1^{-1/2}} - \tilde{k}_2 e^{i\mathbf{p}'\cdot(\mathbf{x}-\gamma\mathbf{x}'/2)\tilde{k}_2^{-1/2}}\right] d\mathbf{y}\, d\mathbf{y}'\, d\mathbf{p}'. \end{aligned} \tag{36}$$

The integral on the right side of (36) is bounded for any $H \in (0, 1), \eta > 0$ or $\eta = 0, H < 1/2$. Hence the function $\mathcal{F}_2^{-1}\overline{\mathcal{K}}_\theta W_s(\mathbf{x}, \mathbf{x}')$ has a compact support and is square-integrable. Similarly, one can show that (25) with (27) and (28) is well defined for $H \in (0, 1), \rho < \infty$ or $H > 1/2, \rho = \infty$.

In the geometrical optics limit,

$$\begin{aligned}\mathcal{F}_2^{-1}\overline{\mathcal{K}}_\theta W_s(\mathbf{x},\mathbf{x}') = (2\pi)^{-d}\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{x}')\int \mathcal{F}_2^{-1}\theta(\mathbf{y},\mathbf{y}')\mathcal{F}_2^{-1}W_z(\mathbf{y},-\mathbf{y}')\\ \times\Bigg[-\tilde{k}\mathbf{x}'\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\mathbf{y}'-\frac{\beta^2}{4\tilde{k}}\mathbf{x}\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\mathbf{y}\\ -\beta^2 D_0(\mathbf{x}-\mathbf{y})+\tilde{k}^{1/2}\beta\mathbf{D}'(\mathbf{x}-\mathbf{y})\cdot i(\mathbf{x}'-\mathbf{y}')\\ -2^{-1}\beta\left[\mathbf{y}\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\mathbf{x}'+\mathbf{x}\cdot\mathbf{D}(\mathbf{x}-\mathbf{y})\cdot\mathbf{y}'\right]\\ +i2^{-1}\tilde{k}^{-1/2}\beta^2(\mathbf{x}-\mathbf{y})\cdot\mathbf{D}'(\mathbf{x}-\mathbf{y})\Bigg]d\mathbf{y}\,d\mathbf{y}'\end{aligned} \tag{37}$$

In view of (35) we see that the right side of (37) has a compact support and is bounded for any $H\in(0,1), \rho<\infty$ or $H>1/2, \rho=\infty$ or $H<1/2, \rho<\infty$.

The evolution equation for the two-frequency mutual coherence function

$$\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2)=\mathbb{E}[\Psi_1(z,\mathbf{x}_1)\Psi_2(z,\mathbf{x}_2)] \tag{38}$$

in the literature[9] can be obtained by setting

$$\mathbf{x} = \frac{1}{2}(\sqrt{\tilde{k}_1}\mathbf{x}_1+\sqrt{\tilde{k}_2}\mathbf{x}_2), \tag{39}$$

$$\mathbf{y} = \frac{1}{\gamma}(\sqrt{\tilde{k}_1}\mathbf{x}_1-\sqrt{\tilde{k}_2}\mathbf{x}_2) \tag{40}$$

and applying $\mathcal{F}_2^{-1}$ to the mean field equation

$$\frac{\partial \bar{W}_z}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}_z = -\overline{\mathcal{Q}_0^*}\bar{W}_z, \tag{41}$$

where

$$\begin{aligned}\overline{\mathcal{Q}_0^*}\bar{W}_z = \int \Phi_\eta^\infty(\mathbf{q})\gamma^{-2}\Big[\tilde{k}_1\tilde{k}_2 e^{-i(\tilde{k}_1^{-1/2}-\tilde{k}_2^{-1/2})\mathbf{q}\cdot\mathbf{x}}\bar{W}_z(\mathbf{x},\mathbf{p}-(\tilde{k}_1^{-1/2}+\tilde{k}_2^{-1/2})\gamma\mathbf{q}/2)\\ +\tilde{k}_1\tilde{k}_2 e^{i(\tilde{k}_1^{-1/2}-\tilde{k}_2^{-1/2})\mathbf{q}\cdot\mathbf{x}}\bar{W}_z(\mathbf{x},\mathbf{p}+(\tilde{k}_1^{-1/2}+\tilde{k}_2^{-1/2})\gamma\mathbf{q}/2)\\ -(\tilde{k}_1^2+\tilde{k}_2^2)\bar{W}_z(\mathbf{x},\mathbf{p})\Big]d\mathbf{q}.\end{aligned}$$

Neither (41) nor the resulting equation for Γ_{12} is exactly solvable and various approximations have been proposed (see refs. 4 and 13 and the references therein).

The mean field equation in the geometrical optics limits (23) and (24) has the universal form

$$\begin{aligned}\frac{\partial \bar{W}_z}{\partial z}+\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}_z &= \tilde{k}\nabla_{\mathbf{p}}\cdot\mathbf{D}(0)\cdot\nabla_{\mathbf{p}}\bar{W}_z(\mathbf{x},\mathbf{p})+i\beta\mathbf{x}\cdot\mathbf{D}(0)\cdot\nabla_{\mathbf{p}}\bar{W}_z(\mathbf{x},\mathbf{p})\\ &\quad -\beta^2 D_0(0)\bar{W}_z(\mathbf{x},\mathbf{p})-\frac{\beta^2}{4\tilde{k}}\mathbf{x}\cdot\mathbf{D}(0)\cdot\mathbf{x}\bar{W}_z(\mathbf{x},\mathbf{p})\\ &= -\tilde{k}\left(-i\nabla_{\mathbf{p}}+\frac{\beta}{2\tilde{k}}\mathbf{x}\right)\cdot\mathbf{D}(0)\cdot\left(-i\nabla_{\mathbf{p}}+\frac{\beta}{2\tilde{k}}\mathbf{x}\right)\bar{W}_z(\mathbf{x},\mathbf{p})\\ &\quad -\beta^2 D_0(0)\bar{W}_z(\mathbf{x},\mathbf{p}),\end{aligned}\tag{42}$$

which is exactly solvable and whose solution is presented in Appendix B.

2.1. White-Noise Models with Large-Scale Inhomogeneities

Our approach is also suitable for the situation where deterministic large-scale inhomogeneities are present. One type of slowly varying, large-scale inhomogeneities is multiplicative and can be modeled by a bounded smooth deterministic function $\mu=\mu(z,\mathbf{x})$ due to variability of any one of the three factors in (5) The second type is additive and can be modeled by adding a smooth background $V_0(z,\mathbf{x})$. Altogether we can treat the random refractive index field of the general type

$$V_0(z,\mathbf{x})+\frac{\mu(z,\mathbf{x})}{\varepsilon}V\left(\frac{z}{\varepsilon^2},\mathbf{x}\right)$$

with a bounded smooth deterministic modulation $\mu(z,\mathbf{x})$ and background $V_0(z,\mathbf{x})$. We describe the results below but omit the details of the argument for simplicity of presentation.

First we consider the case of deterministic, large-scale inhomogeneities of a multiplicative type which has μ, given by (5), as a bounded smooth function$\mu=\mu(z,\mathbf{x})$. The resulting limiting process can be described analogously as above except with the term Φ^∞_η replaced by

$$\begin{aligned}\Phi^\infty_\eta(\mathbf{k}) &\longrightarrow \mu(z,\mathbf{x})\mu(z,\mathbf{y})\Phi^\infty_\eta(\mathbf{k}) \quad \text{in } \overline{\mathcal{Q}},\\ \Phi^\infty_\eta(\mathbf{k}) &\longrightarrow \mu^2(z,\mathbf{x})\Phi^\infty_\eta(\mathbf{k}) \quad \text{in } \overline{\mathcal{Q}}_0.\end{aligned}$$

As a consequence the operator $\overline{\mathcal{Q}}_0$ is no longer of convolution type.

Next we add a slowly varying smooth deterministic background $V_0(z, \mathbf{x})$ to the rapidly fluctuating field $\varepsilon^{-1}\mu(z, \mathbf{x})V(\varepsilon^{-2}z, \mathbf{x})$. Namely we have

$$V_0(z, \mathbf{x}) + \frac{\mu(z, \mathbf{x})}{\varepsilon} V\left(\frac{z}{\varepsilon^2}, \mathbf{x}\right)$$

as the potential term in the parabolic wave equation (4).

The resulting martingale problem has an additional term

$$-\int_0^z \langle W_s, \mathcal{L}_0\theta\rangle \, ds \tag{43}$$

in the martingale formulation, where $\mathcal{L}_0\theta$ has the form

$$\begin{aligned} \mathcal{L}_0\theta(\mathbf{x}, \mathbf{p}) &= -i\gamma^{-1}\mathcal{F}_2\left[\delta_{21}V_0(z, \mathbf{x}, \mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x}, \mathbf{y})\right], \\ \delta_{21}V_0(z, \mathbf{x}, \mathbf{y}) &= \tilde{k}_1 V_0\left(z, \frac{\mathbf{x}}{\sqrt{\tilde{k}_2}} + \frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right) - \tilde{k}_2 V_0\left(z, \frac{\mathbf{x}}{\sqrt{\tilde{k}_1}} - \frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right) \end{aligned} \tag{44}$$

for $\gamma > 0$ fixed in the limit, and the form

$$\begin{aligned} &\mathcal{L}_0\theta(\mathbf{x}, \mathbf{p}) \\ &= -\sqrt{\tilde{k}}\nabla V_0\left(z, \frac{\mathbf{x}}{\sqrt{\tilde{k}}}\right)\cdot \nabla_{\mathbf{p}}\theta(\mathbf{x}, \mathbf{p}) + i\beta\theta(\mathbf{x}, \mathbf{p})\left[V_0\left(z, \frac{\mathbf{x}}{\sqrt{\tilde{k}}}\right) - \frac{\mathbf{x}}{2\sqrt{\tilde{k}}}\cdot\nabla V_0\left(z, \frac{\mathbf{x}}{\sqrt{\tilde{k}}}\right)\right] \end{aligned} \tag{45}$$

in the geometrical optics limit.

2.2. Multiple-Point Correlation Functions

The martingale solutions of the limiting models are uniquely determined by their n-point correlation functions which satisfy a closed set of evolution equations.

Using the function $f(r) = r^n$ in the martingale formulation and taking expectation, we arrive after some algebra the following equation

$$\frac{\partial F^{(n)}}{\partial z} = \sum_{j=1}^{n} \mathbf{p}_j \cdot \nabla_{\mathbf{x}_j} F^{(n)} + \sum_{j=1}^{n} \overline{\mathcal{Q}}_0(\mathbf{x}_j, \mathbf{p}_j)F^{(n)} + \sum_{\substack{j,k=1 \\ j\neq k}}^{n} \overline{\mathcal{Q}}(\mathbf{x}_j, \mathbf{p}_j, \mathbf{x}_k, \mathbf{p}_k)F^{(n)} \tag{46}$$

for the n-point correlation function

$$F^{(n)}(z,\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_n,\mathbf{p}_n)\equiv\mathbb{E}\left[W_z(\mathbf{x}_1,\mathbf{p}_1)\ldots W_z(\mathbf{x}_n,\mathbf{p}_n)\right],$$

where $\overline{\mathcal{Q}}_0(\mathbf{x}_j,\mathbf{p}_j)$ is the operator $\overline{\mathcal{Q}}_0$ acting on the variables $(\mathbf{x}_j,\mathbf{p}_j)$ and $\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)$ is the operator $\overline{\mathcal{Q}}$ acting on the variables $(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)$, namely

$$\begin{aligned}
&\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)F^{(n)}\left(\prod_{i=1}^{n}(\mathbf{x}_i,\mathbf{p}_i)\right)\\
&=\mathbb{E}\left\{\left[\prod_{i\neq j,k}W_z(\mathbf{x}_i,\mathbf{p}_i)\right]\int\Phi_{(\eta,\infty)}(0,\mathbf{q})\gamma^{-2}\right.\\
&\quad\times\left[e^{-i\mathbf{q}\cdot\mathbf{x}_j\tilde{k}_1^{-1/2}}\tilde{k}_1W_z(\mathbf{x}_j,\mathbf{p}_j-\tilde{k}_1^{-1/2}\gamma\mathbf{q}/2)\right.\\
&\qquad\left.-e^{-i\mathbf{q}\cdot\mathbf{x}_j\tilde{k}_2^{-1/2}}\tilde{k}_2W_z(\mathbf{x}_j,\mathbf{p}_j+\tilde{k}_2^{-1/2}\gamma\mathbf{q}/2)\right]\\
&\quad\times\left[e^{i\mathbf{q}\cdot\mathbf{y}_k\tilde{k}_2^{-1/2}}\tilde{k}_2W_z(\mathbf{x}_k,\mathbf{p}_k-\tilde{k}_2^{-1/2}\gamma\mathbf{q}/2)\right.\\
&\qquad\left.\left.-e^{i\mathbf{q}\cdot\mathbf{y}_k\tilde{k}_1^{-1/2}}\tilde{k}_1\ W_z(\mathbf{x}_k,\mathbf{p}_k+\tilde{k}_1^{-1/2}\gamma\mathbf{q}/2)\right]d\mathbf{q}\right\}.
\end{aligned}$$

Equation (46) can be more conveniently written as

$$\frac{\partial F^{(n)}}{\partial z}=\sum_{j=1}^{n}\mathbf{p}_j\cdot\nabla_{\mathbf{x}_j}F^{(n)}+\sum_{j,k=1}^{n}\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)F^{(n)}\tag{47}$$

with the identification $\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_j,\mathbf{p}_j)=\overline{\mathcal{Q}}_0(\mathbf{x}_j,\mathbf{p}_j)$. The operator

$$\mathcal{Q}_{\mathrm{sum}}=\sum_{j,k=1}^{n}\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)\tag{48}$$

is a non-positive symmetric operator.

The uniqueness for Eq. (46) with any initial data

$$F^{(n)}(z=0,\mathbf{x}_1,\mathbf{p}_1,\ldots,\mathbf{x}_n,\mathbf{p}_n)=\mathbb{E}\left[W_0(\mathbf{x}_1,\mathbf{p}_1)\ldots W_0(\mathbf{x}_n,\mathbf{p}_n)\right],\quad W_0\in L^2(\mathbb{R}^{2d})$$

in the case of the Wigner–Moyal equation can be easily established by observing that the operator given by (48) is self-adjoint. For instance, for $n=2$, we have that

$$\mathcal{F}_2^{-1}\overline{\mathcal{Q}}F^{(2)}(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2)=\mathcal{F}_2^{-1}\overline{\mathcal{Q}}(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2)\mathcal{F}_2^{-1}F^{(2)}(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2),$$

where

$$\begin{aligned}\mathcal{F}_2^{-1}F^{(2)}(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2) &= \mathbb{E}\left[\mathcal{F}_2^{-1}W_z(\mathbf{x}_1,\mathbf{y}_1)\mathcal{F}_2^{-1}W_z(\mathbf{x}_2,\mathbf{y}_2)\right]\\ \mathcal{F}_2^{-1}\overline{\mathcal{Q}}(\mathbf{x}_1,\mathbf{y}_1,\mathbf{x}_2,\mathbf{y}_2) &= \int d\mathbf{q}\,\Phi_{(\eta,\infty)}(0,\mathbf{q})\gamma^{-2}\Big[\tilde{k}_1 e^{i\mathbf{q}\cdot(\mathbf{x}_1-\gamma\mathbf{y}_1/2)\tilde{k}_1^{-1/2}}\\ &\qquad -\tilde{k}_2 e^{i\mathbf{q}\cdot(\mathbf{x}_1+\gamma\mathbf{y}_1/2)\tilde{k}_2^{-1/2}}\Big]\\ &\qquad \times\left[\tilde{k}_2 e^{-i\mathbf{q}\cdot(\mathbf{x}_2+\gamma\mathbf{y}_2/2)\tilde{k}_2^{-1/2}} - \tilde{k}_1 e^{-i\mathbf{q}\cdot(\mathbf{x}_2-\gamma\mathbf{y}_2/2)\tilde{k}_1^{-1/2}}\right].\end{aligned}$$

Namely, in the $(\mathbf{x}_j,\mathbf{y}_j)$ variables, the operator $\mathcal{Q}_{\text{sum}}$ becomes the multiplication by a function which is dominated by the "diagonal terms" with $j=k$

$$\begin{aligned}\mathcal{F}_2^{-1}\overline{\mathcal{Q}}_0(\mathbf{x}_j,\mathbf{y}_j) = -\int \Phi_{(\eta,\infty)}(0,\mathbf{q})\gamma^{-2}\Big|\tilde{k}_1 e^{i\mathbf{q}\cdot(\mathbf{x}_j-\gamma\mathbf{y}_j/2)\tilde{k}_1^{-1/2}}\\ - \tilde{k}_2 e^{i\mathbf{q}\cdot(\mathbf{x}_j+\gamma\mathbf{y}_j/2)\tilde{k}_2^{-1/2}}\Big|^2\,d\mathbf{q}\end{aligned}\tag{49}$$

and hence is non-positive. Therefore $\mathcal{Q}_{\text{sum}}$ is a non-positive self-adjoint operator on L^2. The case with $n>2$ is similar.

Each of the two operators on the right side of (47) generates a unique C_0-semigroup of contractions on $L^2(\mathbb{R}^{2nd})$ and, by the product formula, their sum generates a unique C_0-semigroup of contractions on $L^2(\mathbb{R}^{2nd})$. Standard theory for linear equations then yields the uniqueness result for the weak solution of (47).

In the geometrical optics limit $\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{p}_j,\mathbf{x}_k,\mathbf{p}_k)$ in Eq. (47) takes the form

$$\begin{aligned}\mathcal{F}_2^{-1}\overline{\mathcal{Q}}(\mathbf{x}_j,\mathbf{y}_j,\mathbf{x}_k,\mathbf{y}_k) &= -\tilde{k}\mathbf{y}_j\cdot\mathbf{D}(\mathbf{x}_j-\mathbf{x}_k)\cdot\mathbf{y}_k - \frac{\beta^2}{4\tilde{k}}\mathbf{x}_j\cdot\mathbf{D}(\mathbf{x}_j-\mathbf{x}_k)\cdot\mathbf{x}_k\\ &\quad -\beta^2 D_0(\mathbf{x}_j-\mathbf{x}_k)+\tilde{k}^{1/2}\beta\mathbf{D}'(\mathbf{x}_j-\mathbf{x}_k)\cdot i(\mathbf{y}_j-\mathbf{y}_k)\\ &\quad -2^{-1}\beta\left[\mathbf{x}_k\cdot\mathbf{D}(\mathbf{x}_j-\mathbf{x}_k)\cdot\mathbf{y}_j+\mathbf{x}_j\cdot\mathbf{D}(\mathbf{x}_j-\mathbf{x}_k)\cdot\mathbf{y}_k\right]\\ &\quad +i2^{-1}\tilde{k}^{-1/2}\beta^2(\mathbf{x}_j-\mathbf{x}_k)\cdot\mathbf{D}'(\mathbf{x}_j-\mathbf{x}_k).\end{aligned}$$

The uniqueness follows from the same argument as in the previous case.

3. FORMULATION AND MAIN THEOREMS

3.1. Martingale Formulation

The tightness result (see below) implies that for L^2 initial data the limiting measure $\mathbb{P}$ is supported in $C([0,\infty); L^2(\mathbb{R}^{2d}))$. For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Consider a special class of admissible functions $f_z = f(\langle W_z^\varepsilon, \theta\rangle)$, $f_z' = f'(\langle W_z^\varepsilon, \theta\rangle)$, $\forall f \in C^\infty(\mathbb{R})$. We have the following expression

$$\mathcal{A}^\varepsilon f_z = f_z' \left[\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle + \frac{1}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon *}\theta\rangle \right]. \tag{50}$$

A main property of $\mathcal{A}^\varepsilon$ is that

$$f_z - \int_0^z \mathcal{A}^\varepsilon f_s\, ds \quad \text{is a } \mathcal{F}_z^\varepsilon\text{-martingale}, \quad \forall f \in \mathcal{D}(\mathcal{A}^\varepsilon). \tag{51}$$

Let $\mathcal{F}_z^\varepsilon$ be the σ-algebras generated by $\{V_s^\varepsilon,\ s \leqslant t\}$ and $\mathbb{E}_z^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathcal{F}_z^\varepsilon$. Then we also have

$$\mathbb{E}_s^\varepsilon f_z - f_s = \int_s^z \mathbb{E}_s^\varepsilon \mathcal{A}^\varepsilon f_\tau\, d\tau \quad \forall s < z \ \text{ a.s.} \tag{52}$$

(see ref. 10). Note that the process W_z^ε is not Markovian and $\mathcal{A}^\varepsilon$ is not its generator. We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $V_z(\cdot) = V(z,\cdot)$.

3.2. Assumptions and Properties of the Refractive Index Field

As mentioned in the introduction, we assume that $V_z(\mathbf{x})$ is a square-integrable, z-stationary, $\mathbf{x}$-homogeneous process with a spectral density satisfying the upper bound (6).

Let $\mathcal{F}_z$ and $\mathcal{F}_z^+$ be the sigma-algebras generated by $\{V_s : \forall s \leqslant z\}$ and $\{V_s : \forall s \geqslant z\}$, respectively. The correlation coefficient $r_{\eta,\rho}(t)$ is given by

$$r_{\eta,\rho}(t) = \sup_{\substack{h\in\mathcal{F}_z \\ \mathbb{E}[h]=0,\mathbb{E}[h^2]=1}} \ \sup_{\substack{g\in\mathcal{F}_{z+t}^+ \\ \mathbb{E}[g]=0,\mathbb{E}[g^2]=1}} \mathbb{E}[hg]. \tag{53}$$

Lemma 1. The correlation coefficient $r_{\eta,\rho}(t)$ satisfies the inequality

$$\begin{aligned}|\mathbb{E}[\mathbb{E}_z[V_s(\mathbf{x})]\mathbb{E}_z[V_t(\mathbf{y})]]| &= |\mathbb{E}[\mathbb{E}_z[V_s(\mathbf{x})]V_t(\mathbf{y})]| \\ &\leqslant r_{\eta,\rho}(s-z)r_{\eta,\rho}(t-z)\mathbb{E}\left[V_z^2\right], \quad \forall s,t \geqslant z, \ \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^d.\end{aligned}$$

Proof. Let

$$h_s(\mathbf{x}) = \mathbb{E}_z[V_s(\mathbf{x})], \quad g_t(\mathbf{x}) = V_t(\mathbf{x}).$$

Clearly

$$\begin{aligned} h_s &\in L^2(P,\Omega,\mathcal{F}_z), \\ g_t &\in \in L^2(P,\Omega,\mathcal{F}_t^+) \end{aligned}$$

and their second moments are uniformly bounded in $\mathbf{x}$ since

$$\begin{aligned} \mathbb{E}[h_s^2](\mathbf{x}) &\leqslant \mathbb{E}[g_s^2](\mathbf{x}), \\ \mathbb{E}[g_s^2](\mathbf{x}) &= \int \Phi(\xi,\mathbf{q})d\xi\, d\mathbf{q}. \end{aligned}$$

From Definition (53) we have

$$|\mathbb{E}[h_s(\mathbf{x})h_t(\mathbf{y})]| = |\mathbb{E}[h_s g_t]| \leqslant r_{\eta,\rho}(t-z)\mathbb{E}^{1/2}\left[h_s^2(\mathbf{x})\right]\mathbb{E}^{1/2}\left[g_t^2\right].$$

Hence by setting $s=t$ first and the Cauchy–Schwartz inequality we have

$$\begin{aligned} \mathbb{E}[h_s^2(\mathbf{x})] &\leqslant r_{\eta,\rho}^2(s-z)\mathbb{E}[g_s^2], \\ \mathbb{E}[h_s(\mathbf{x})h_t(\mathbf{y})] &\leqslant r_{\eta,\rho}(t-z)r_{\eta,\rho}(s-z)\mathbb{E}[g_t^2], \quad \forall s,t \geqslant z, \forall \mathbf{x},\mathbf{y}. \quad \blacksquare \end{aligned}$$

We assume

Assumption 1. The correlation coefficient $r_{\eta,\rho}(t)$ satisfies

$$\int_0^\infty r_{\eta,\rho}(s)ds < \infty.$$

Corollary 1. The formula

$$\tilde{V}_z(\mathbf{x}) = \int_z^\infty \mathbb{E}_z\left[V_s(\mathbf{x})\right]\, ds \tag{54}$$

defines a square-integrable z-stationary, $\mathbf{x}$-homogeneous process.

Proof. Let $\omega \in \Omega$ denote the random element and $\tau_{\vec{\mathbf{x}}}, \vec{\mathbf{x}} = (z, \mathbf{x}) \in \mathbb{R}^{d+1}$ the translation operator acting on Ω. Then without loss of generality we may assume that there exists a square-integrable function V defined on Ω such that

$$V_z(\mathbf{x}, \omega) = V(\tau_{\vec{\mathbf{x}}}\omega).$$

It suffices to show that the second moment of

$$\tilde{V}(\omega) \equiv \int_0^\infty \mathbb{E}_0\left[V(\tau_{(s,0)}\omega)\right]\, ds$$

is finite since

$$\tilde{V}_z(\mathbf{x}, \omega) = \tilde{V}(\tau_{\vec{\mathbf{x}}}\omega), \qquad \forall \vec{\mathbf{x}} = (z, \mathbf{x}) \in \mathbb{R}^{d+1}.$$

To this end we have

$$\begin{aligned}
\mathbb{E}\left[\tilde{V}^2\right] &= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{E}_0[V_s(0)]\mathbb{E}_0[V_t(0)] ds\, dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \int_0^\infty \mathbb{E}_0[V_s(0)] V_t(0) ds\, dt\right] \\
&\leqslant \int_0^\infty \int_0^\infty r_{\eta,\rho}(s) r_{\eta,\rho}(t) ds\, dt \mathbb{E}[V_0^2],
\end{aligned}$$

which is finite by Assumption 1. ▮

In the Gaussian case the correlation coefficient $r_{\eta,\rho}(t)$ equals the linear correlation coefficient given by

$$r_{\eta,\rho}(t) = \sup_{g_1, g_2} \int R(t - \tau_1 - \tau_2, \mathbf{k}) g_1(\tau_1, \mathbf{k}) g_2(\tau_2, \mathbf{k}) d\mathbf{k}\, d\tau_1\, d\tau_2, \tag{55}$$

where

$$R(t,\mathbf{k}) = \int e^{it\xi}\,\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi$$

and the supremum is taken over all $g_1, g_2 \in L^2(\mathbb{R}^{d+1})$, which are supported on $(-\infty, 0] \times \mathbb{R}^d$ and satisfy the constraint

$$\int R(t-t',\mathbf{k})g_1(t,\mathbf{k})g_1^*(t',\mathbf{k})dtdt'd\mathbf{k} = \int R(t-t',\mathbf{k})g_2(t,\mathbf{k})g_2^*(t',\mathbf{k})dt\,dt'\,d\mathbf{k} = 1, \tag{56}$$

in ref. 8. Alternatively, by the Paley–Wiener theorem we can write

$$r_{\eta,\rho}(t) = \sup_{f_1,f_2} \int e^{i\xi t} f_1(\xi,\mathbf{k}) f_2(\xi,\mathbf{k})\Phi_{\eta,\rho}(\xi,\mathbf{k})d\xi\,d\mathbf{k}, \tag{57}$$

where f_1, f_2 are elements of the Hardy space $\mathcal{H}^2$ of $L^2(\Phi_{(\eta,\rho)}d\xi d\mathbf{k})$-valued analytic functions in the upper half ξ-space satisfying the normalization condition

$$\int |f_j(\xi,\mathbf{k})|^2\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} = 1, \quad j = 1, 2.$$

There are various criteria for the decay rate of the linear correlation coefficients, see ref. 8.

Corollary 2. If V_z is a Gaussian random field and its linear correlation coefficient $r_{\eta,\rho}(t)$ is integrable, then $\tilde{V}_z$ is also Gaussian and hence possesses finite moments of all orders.

This follows from the fact that the mapping from V_z to $\tilde{V}_z$ is a bounded linear operator on the Gaussian space.

The main property of $\tilde{V}_z$ as a random function is that

$$\mathcal{A}\tilde{V}_z = -V_z, \quad \text{a.s.} \quad z \in \mathbb{R}. \tag{58}$$

Since $\mathcal{A}$ commutes with the shift in $\mathbf{x}$ so the appearance of $\mathbf{x}$ in Eq. (58) is suppressed.

We have the following simple relation

$$\begin{aligned}\lim_{\lambda\to\infty} \mathbb{E}\left[\tilde{V}_{z\lambda}(\mathbf{x})V_{z\lambda}(\mathbf{y})\right] &= \lim_{\lambda\to\infty} \int e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}} \int \frac{1}{i\xi}\left(e^{iz\lambda\xi} - 1\right)\Phi_{(\eta,\rho)}(\xi,\mathbf{p})d\xi\,d\mathbf{p} \\ &= \pi \int e^{i(\mathbf{x}-\mathbf{y})\cdot\mathbf{p}}\Phi_{(\eta,\rho)}(0,\mathbf{p})d\mathbf{p}, \quad \forall z.\end{aligned} \tag{59}$$

Assumption 2. For any $\eta > 0$,

$$R_\eta = \limsup_{\rho\to\infty} \int_0^\infty r_{\eta,\rho}(t)dt < \infty$$

such that

$$\limsup_{\eta\to 0} \eta R_\eta < \infty.$$

For Gaussian fields with the generalized von Kármán spectrum[12]

$$\Phi_{vk}(\vec{\mathbf{k}}) = 2^{H-1}\Gamma\left(H + \frac{d+1}{2}\right)\eta^{2H}\pi^{-(d+1)/2}(\eta^2 + |\vec{\mathbf{k}}|^2)^{-H-1/2-d/2} \quad (60)$$

a straightforward scaling argument with (57) shows that

$$r_{\eta,\infty}(t) = r_{1,\infty}(\eta t)$$

hence

$$R_\eta = \eta^{-1} R_1.$$

This motivates Assumption 2. Set

$$\tilde{\Phi}_z^\varepsilon(\mathbf{k}) \equiv \tilde{\Phi}_{\varepsilon^{-2}z}(\xi, \mathbf{k}),$$

which is the spectral density of $\tilde{V}_z^\varepsilon(\mathbf{x}) \equiv \tilde{V}_{z/\varepsilon^2}(\mathbf{x})$.

Define analogously to (19)

$$\tilde{\mathcal{L}}_z^{\varepsilon *}\theta(\mathbf{x}, \mathbf{p}) \equiv -i\gamma^{-1}\mathcal{F}_2\left[\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x}, \mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x}, \mathbf{y})\right] \quad (61)$$

with

$$\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x}, \mathbf{y}) \equiv \tilde{k}_1\tilde{V}_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}_2}} + \frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_2}}\right) - \tilde{k}_2\tilde{V}_z^\varepsilon\left(\frac{\mathbf{x}}{\sqrt{\tilde{k}_1}} - \frac{\gamma\mathbf{y}}{2\sqrt{\tilde{k}_1}}\right). \quad (62)$$

We also need to know the first few moments the random fields involved. The case of Gaussian fields motivates the following assumption of the sixth order sub-Gaussian property.

Assumption 3.

$$\sup_{|\mathbf{y}|\leqslant L} \mathbb{E}\left[\delta_{21} V_z^\varepsilon(\mathbf{y})\right]^4 \leqslant C_1 \sup_{|\mathbf{y}|\leqslant L} \mathbb{E}^2\left[\delta_{21} V_z^\varepsilon\right]^2(\mathbf{y}), \tag{63}$$

$$\sup_{|\mathbf{y}|\leqslant L} \mathbb{E}\left[\delta_{21} \tilde{V}_z^\varepsilon\right]^4(\mathbf{y}) \leqslant C_2 \sup_{|\mathbf{y}|\leqslant L} \mathbb{E}^2\left[\delta_{21} \tilde{V}_z^\varepsilon\right]^2(\mathbf{y}), \tag{64}$$

$$\sup_{|\mathbf{y}|\leqslant L} \mathbb{E}\left[\left[\delta_{21} V_z^\varepsilon\right]^2\left[\delta_{21} \tilde{V}_z^\varepsilon\right]^4\right](\mathbf{y}) \leqslant C_3 \left(\sup_{|\mathbf{y}|\leqslant L} \mathbb{E}\left[\delta_{21} V_z^\varepsilon\right]^2(\mathbf{y})\right) \times \left(\sup_{|\mathbf{y}|\leqslant L} \mathbb{E}^2\left[\delta_{21} \tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right) \tag{65}$$

for all $L<\infty$, where the constants C_1, C_2 and C_3 are independent of $\varepsilon, \eta, \rho, \gamma$.

From (19) and (61) we can form the iteration of operators $\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^{\varepsilon *}$

$$\mathcal{L}_z^{\varepsilon *} \tilde{\mathcal{L}}_z^{\varepsilon *}\theta(\mathbf{x},\mathbf{p}) = -\gamma^{-2}\mathcal{F}_2\left[\delta_{21} V_z^\varepsilon(\mathbf{x},\mathbf{y})\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right],$$

The operator $\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^{\varepsilon *}\theta$ is well-defined if $\delta_{21} V_z^\varepsilon$ and $\delta_{21}\tilde{V}_z^\varepsilon$ are locally square-integrable. Other iterations of $\mathcal{L}_z^\varepsilon$ and $\tilde{\mathcal{L}}_z^{\varepsilon *}$ allowed by Assumption 3 can be similarly constructed.

Assumption 4. For every $\theta\in\mathcal{S}$, there exists a random constant C_5 such that

$$\sup_{z<z_0} \|\delta_{21}\tilde{V}_z^\varepsilon \mathcal{F}_2^{-1}\theta\|_4 \leqslant \frac{C_5}{\sqrt{\varepsilon}} \sup_{\substack{z\in[0,z_0]\\ |\mathbf{x}|,|\mathbf{y}|\leqslant L}} \mathbb{E}^{1/2}|\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})|^2, \quad \forall\theta\in\mathcal{S}, \varepsilon, \eta, \quad \gamma\leqslant 1\leqslant\rho \tag{66}$$

with C_5 possessing finite moments and depending only on θ, z_0, where L is the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$. cf. Lemma 2 and (67).

For a Gaussian random field, Assumption 4 is readily satisfied by Lemma 2 and Borell's inequality[1]

$$\begin{aligned}\sup_{z<z_0} \|\delta_{21}\tilde{V}_z^\varepsilon \mathcal{F}_2^{-1}\theta\|_4 &\leqslant \|\mathcal{F}_2^{-1}\theta\|_4 \sup_{\substack{z\in[0,z_0]\\ |\mathbf{x}|,|\mathbf{y}|\leqslant L}} |\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})| \\ &\leqslant C_5 \log\left(\frac{z_0}{\varepsilon^2}\right) \sup_{\substack{z\in[0,z_0]\\ |\mathbf{x}|,|\mathbf{y}|\leqslant L}} \mathbb{E}^{1/2}|\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})|^2, \\ &\quad \forall\eta, \gamma\leqslant 1\leqslant\rho,\end{aligned} \tag{67}$$

where the random constants C_5 has a Gaussian-like tail.

Note that with γ or ρ held fixed the first term on the right side of (66) is always $O(1)$. Compared to the corresponding condition (67) for the Gaussian field condition (66) allows for certain degree of intermittency in the refractive index field.

As we have seen above, most of the assumptions here are motivated by the Gaussian case and we have formulated them in such a way as to allow some level of non-Gaussian fluctuation.

3.3. Main Theorems

Theorem 1. Let V_z^ε be a z-stationary, $\mathbf{x}$-homogeneous, almost surely locally bounded random process with the spectral density satisfying the bound (6) and Assumptions 1–4. Let $\gamma > 0$ be fixed.

(i) Let η be fixed and ρ be fixed or tend to ∞ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \varepsilon\rho^{2-H} = 0. \tag{68}$$

Then any weak solutions $W^\varepsilon \in C([0,\infty); L^2(\mathbb{R}^{2d})]$ of the Wigner–Moyal equation with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ and $\|W_z^\varepsilon\|_2 \leqslant \|W_0\|_2, \forall z > 0$ converge in probability in the space $C([0,\infty); L_w^2(\mathbb{R}^{2d}))$ to that of the corresponding Gaussian white-noise model with the covariance operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}_0$ as given by (25) and (26), respectively (see also (43) and (44)). The statement holds true for any $H \in (0,1)$.

(ii) Suppose additionally that $H < 1/2$ and $\eta = \eta(\varepsilon) \to 0$ such that

$$\lim_{\varepsilon\to 0} \varepsilon\eta^{-1}(\eta^{-1} + \rho^{2-H}) = 0. \tag{69}$$

Then the same convergence holds true.

Theorem 2 concerns a similar convergence to the solution of a Gaussian white-noise model for the Liouville equation.

Theorem 2. Let V_z^ε be a z-stationary, $\mathbf{x}$-homogeneous, almost surely smooth, locally bounded random process with the spectral density satisfying the bound (6) and Assumptions 1–4.

Let $\gamma = \gamma(\varepsilon) \to 0, \tilde{k}_1, \tilde{k}_2 \to \tilde{k}$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \gamma^{-1}(\tilde{k}_2 - \tilde{k}_1) = \beta$$

for some finite, positive constant β. Then under any of the following three sets of conditions

(i) $\rho<\infty$ and $\eta>0$ held fixed;

(ii) $H>1/2$, $\eta>0$ fixed and $\rho=\rho(\varepsilon)\to\infty$ as $\varepsilon\to 0$ such that

$$\lim_{\varepsilon\to 0}\varepsilon\rho^{2-H}=0; \tag{70}$$

(iii) $H<1/2$, $\rho<\infty$ fixed and $\eta=\eta(\varepsilon)\to 0$ such that

$$\lim_{\varepsilon\to 0}\varepsilon\eta^{-2}=0 \tag{71}$$

any weak solutions $W^\varepsilon\in C([0,\infty);L^2(\mathbb{R}^{2d})]$ of the Wigner–Moyal equation with the initial condition $W_0\in L^2(\mathbb{R}^{2d})$ and $\|W_z^\varepsilon\|_2\leqslant\|W_0\|_2, \forall z>0$ converge in probability in the space $C([0,\infty);L^2_w(\mathbb{R}^{2d}))$ to the martingale solution of the Gaussian white-noise model with the covariance operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{Q}}_0$ as given by (27) and (28), respectively (see also (43) and (45)).

Remark 1. Because $\left\langle W_z^\varepsilon,\theta\right\rangle$ is uniformly bounded by $\|W_0\|_2\|\theta\|_2$ we have from the above convergence theorems the convergence of moments, namely for any $0\leqslant z_1\leqslant z_2\leqslant\cdots\leqslant z_n$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\left\{\left\langle W_{z_1}^\varepsilon,\theta\right\rangle\cdots\left\langle W_{z_n}^\varepsilon,\theta\right\rangle\right\}=\mathbb{E}\left\{\left\langle W_{z_1},\theta\right\rangle\cdots\left\langle W_{z_n},\theta\right\rangle\right\}.$$

Note that the Kolmogorov value $H=1/3$ is covered by the regimes of Theorems 1 and 2(i), (iii).

4. PROOF OF THEOREM 1 AND 2

The argument is similar to that for the one-frequency setting in ref. 5. We reproduce it here with minor adaption to the two-frequency setting for the sake of completeness and the convenience of the reader.

First we establish some technical results for the proof of the theorems.

Lemma 2. (Appendix A) For each $z_0<\infty$ there exists a positive constant $\tilde{C}<\infty$ such that

$$\sup_{\substack{|z|\leqslant z_0\\ |\mathbf{y}|\leqslant L}}\mathbb{E}\left[\left(\delta_{21}V_z^\varepsilon\right)^2\right](\mathbf{y})\leqslant\tilde{C}\gamma^2\left|\min\left(\gamma^{-1},\rho\right)\right|^{2-2H}$$

$$\sup_{|z|\leqslant z_0}\mathbb{E}\left[\tilde{V}_z^\varepsilon(\mathbf{x})\right]^2\leqslant\tilde{C}\eta^{-2-2H}$$

$$\sup_{\substack{|z|\leqslant z_0\\|\mathbf{y}|\leqslant L}} \mathbb{E}\left[\left(\delta_{21}\tilde{V}_z^{\varepsilon}\right)^2\right](\mathbf{y}) \leqslant \tilde{C}\eta^{-2}\gamma^2|\min(\rho,\gamma^{-1})|^{2-2H}$$

$$\sup_{\substack{|z|\leqslant z_0\\|\mathbf{y}|\leqslant L}} \left|\nabla_{\mathbf{y}}\mathbb{E}\left[\delta_{21}\tilde{V}_z^{\varepsilon}\right]^2(\mathbf{y})\right| \leqslant \tilde{C}\eta^{-2}\gamma^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}$$

$$\sup_{|z|\leqslant z_0} \mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^{\varepsilon*}\theta)\|_2^2 \leqslant \tilde{C}\eta^{-2}\rho^{4-2H}, \quad \theta\in\mathcal{S}$$

for all $H\in(0,1), \varepsilon,\gamma,\eta\leqslant 1\leqslant\rho, \mathbf{x},\mathbf{y}\in\mathbb{R}^d$, where the constant $\tilde{C}$ depends only on z_0, L and θ.

The following estimates can be obtained from Lemma 2 and Assumption 3.

Corollary 3.

$$\begin{aligned}\mathbb{E}\left[\|\mathcal{L}_z^{\varepsilon*}\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^{\varepsilon*}\theta(\mathbf{y},\mathbf{q})\|_2^2\right] &\leqslant C\left(\eta^{-2}|\min(\rho,\gamma^{-1})|^{4-4H}\right),\\ \mathbb{E}\left[\|\mathcal{L}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\|_2^2\right] &\leqslant C\left(\eta^{-2}|\min(\rho,\gamma^{-1})|^{4-4H}\right),\\ \mathbb{E}\left[\|\tilde{\mathcal{L}}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\|_2^2\right] &\leqslant C\left(\eta^{-4}|\min(\rho,\gamma^{-1})|^{4-4H}\right),\\ \mathbb{E}\left\|\mathcal{L}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\|_2^2 &\leqslant C\left(\eta^{-4}|\min(\rho,\gamma^{-1})|^{6-6H}\right),\end{aligned}$$

where the constant C is independent of ρ,η,γ and L is the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$.

4.1. Tightness

In the sequel we will adopt the following notation

$$f_z\equiv f(\langle W_z^{\varepsilon},\theta\rangle),\quad f_z'\equiv f'(\langle W_z^{\varepsilon},\theta\rangle),\quad f_z''\equiv f''(\langle W_z^{\varepsilon},\theta\rangle),\\ \forall f\in C^{\infty}(\mathbb{R}). \tag{72}$$

Namely, the prime stands for the differentiation w.r.t. the original argument (not z) of f, f' etc. Let L denote the radius of the ball containing the support of $\mathcal{F}_2^{-1}\theta$. Let all the constants $c, c', c_1, c_2, \ldots$ etc., in the sequel be independent of ρ,η,γ and ε and depend only on $z_0,\theta,\|W_0\|_2$ and f.

First we note that since $\mathcal{S}$ is dense in $L^2(\mathbb{R}^{2d})$ and $\|W_z^{\varepsilon}\|_2\leqslant\|W_0\|_2, \forall z>0$, the tightness of the family of $L^2(\mathbb{R}^{2d})$-valued processes $\{W^{\varepsilon}, 0<\varepsilon<1\}$ in the Skorohod space $D([0,\infty); L_w^2(\mathbb{R}^{2d})$ is equivalent to the tightness of the family in the Skorohod space $D([0,\infty);\mathcal{S}')$ as distribution-valued processes. According to ref. 7, a family of processes $\{W^\varepsilon, 0<\varepsilon<1\}\subset D([0,\infty);\mathcal{S}')$ is tight if and only if for every test function $\theta\in\mathcal{S}$ the family of processes $\{\langle W^\varepsilon,\theta\rangle, 0<\varepsilon<1\}\subset D([0,\infty);\mathbb{R})$ is tight. With this remark we can now use the tightness criterion of ref. 11. (Chap. 3, Theorem 4) for finite dimensional processes, namely, we will prove: First,

$$\lim_{N\to\infty}\limsup_{\varepsilon\to 0}\mathbb{P}\{\sup_{z<z_0}|\left\langle W_z^\varepsilon,\theta\right\rangle|\geqslant N\}=0,\quad \forall z_0<\infty. \tag{73}$$

Second, for each $f\in C^\infty(\mathbb{R})$ there is a sequence $f_z^\varepsilon\in\mathcal{D}(\mathcal{A}^\varepsilon)$ such that for each $z_0<\infty\{\mathcal{A}^\varepsilon f_z^\varepsilon, 0<\varepsilon<1, 0<z<z_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0}\mathbb{P}\{\sup_{z<z_0}|f_z^\varepsilon-f(\left\langle W_z^\varepsilon,\theta\right\rangle)|\geqslant\delta\}=0,\quad\forall\delta>0. \tag{74}$$

Then it follows that the laws of $\{\langle W^\varepsilon,\theta\rangle, 0<\varepsilon<1\}$ are tight in the space of $D([0,\infty);\mathbb{R})$ and hence $\{W_z^\varepsilon\}$ is tight in $D([0,\infty);L_w^2(\mathbb{R}^{2d}))$.

To prove the tightness in the space $C([0,\infty);L_w^2(\mathbb{R}^{2d})$ we only need to note first that $W_z^\varepsilon\in C([0,\infty);L_w^2(\mathbb{R}^{2d})$ (see the construction of solution) and second that the Skorohod metric and the uniform metric induce the same topology on $C([0,\infty);L_w^2(\mathbb{R}^{2d}))$.

Condition (73) is satisfied because the L^2-norm is preserved.

We shall construct a test function of the form $f_z^\varepsilon=f_z+f_{1,z}^\varepsilon+f_{2,z}^\varepsilon+f_{3,z}^\varepsilon$. First we construct the first perturbation $f_{1,z}^\varepsilon$. Let

$$\tilde{V}_z^\varepsilon=\tilde{V}_{z/\varepsilon^2}.$$

Recall that

$$\mathcal{A}^\varepsilon\tilde{V}_z^\varepsilon=-\varepsilon^{-2}V_z^\varepsilon.$$

Define the first perturbation as

$$f_{1,z}^\varepsilon\equiv\frac{1}{\varepsilon}\int_z^\infty f_z'\left\langle W_z^\varepsilon,\mathbb{E}_z^\varepsilon\mathcal{L}_s^{\varepsilon *}\theta\right\rangle ds. \tag{75}$$

We have

$$\begin{aligned}
f_{1,z}^\varepsilon&=\varepsilon f_z'\left\langle\mathcal{F}_2^{-1}W_z^\varepsilon,\gamma^{-1}\delta_{21}\int_z^\infty\mathbb{E}_z[V_s^\varepsilon]ds\,\mathcal{F}_2^{-1}\theta\right\rangle\\
&=\varepsilon f_z'\left\langle\mathcal{F}_2^{-1}W_z^\varepsilon,\gamma^{-1}\delta_{21}\tilde{V}_z^\varepsilon\mathcal{F}_2^{-1}\theta\right\rangle\\
&=\varepsilon f_z'\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle.
\end{aligned}$$

Proposition 1.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f^{\varepsilon}_{1,z}|=0,\quad \lim_{\varepsilon\to 0}\sup_{z<z_0}|f^{\varepsilon}_{1,z}|=0\quad\text{in probability.}$$

Proof. First

$$\begin{aligned}\mathbb{E}[|f^{\varepsilon}_{1,z}|] &\leqslant \varepsilon\|f'\|_\infty\|W_0\|_2\mathbb{E}\|\tilde{\mathcal{L}}^{\varepsilon}_z\theta\|_2 \qquad (76)\\ &\leqslant c\varepsilon\|f'\|_\infty\|W_0\|_2\sup_{|\mathbf{x}|,|\mathbf{y}|\leqslant L}\mathbb{E}^{1/2}\left[\gamma^{-1}\delta_{21}\tilde{V}^{\varepsilon}_z(\mathbf{x},\mathbf{y})\right]^2\\ &= O\left(\varepsilon\eta^{-1}|\min(\rho,\gamma^{-1})|^{1-H}\right), \qquad (77)\end{aligned}$$

which is of the following order of magnitude:

$$\begin{cases}\varepsilon & \text{if } \eta,\rho \text{ held fixed,}\\ \varepsilon & \text{if } \gamma,\eta \text{ held fixed,}\\ \varepsilon\eta^{-1} & \text{if } \gamma \text{ or } \rho \text{ held fixed,}\\ \varepsilon|\min(\rho,\gamma^{-1})|^{1-H} & \text{if } \eta \text{ is held fixed}\end{cases} \qquad (78)$$

and vanishes in the respective regimes. Second, we have

$$\begin{aligned}\sup_{z<z_0}|f^{\varepsilon}_{1,z}| &\leqslant \varepsilon\|f'\|_\infty\|W_0\|_2\sup_{z<z_0}\gamma^{-1}\|\delta_{21}\tilde{V}^{\varepsilon}_z\mathcal{F}_2^{-1}\theta\|_2\\ &\leqslant c\varepsilon^{1/2}\sup_{|\mathbf{x}|,|\mathbf{y}|\leqslant L}\mathbb{E}^{1/2}|\gamma^{-1}\delta_{21}\tilde{V}^{\varepsilon}_z(\mathbf{x},\mathbf{y})|^2\\ &= c'\varepsilon^{1/2}\eta^{-1}|\min(\rho,\gamma^{-1})|^{1-H} \qquad (79)\end{aligned}$$

by Assumption 4, with a random constant c' possessing finite moments. On the right side of (79) is of the following order of magnitude:

$$\begin{cases}\varepsilon^{1/2} & \text{if } \eta,\rho \text{ held fixed,}\\ \varepsilon^{1/2} & \text{if } \gamma,\eta \text{ held fixed,}\\ \varepsilon^{1/2}\eta^{-1} & \text{if } \rho \text{ or } \gamma \text{ held fixed,}\\ \varepsilon^{1/2}|\min(\rho,\gamma^{-1})|^{1-H} & \text{if } \eta \text{ is held fixed,}\end{cases} \qquad (80)$$

which vanishes in the respective regimes. On The right side of (79) now converges to zero in probability by a simple application of Chebyshev's inequality and (69). ∎

A straightforward calculation yields

$$\mathcal{A}^\varepsilon f_1^\varepsilon = -\varepsilon f_z'\left\langle W_z^\varepsilon, \left[\mathbf{p}\cdot\nabla + \frac{1}{\varepsilon}\mathcal{L}_z^{\varepsilon*}\right]\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle - \frac{1}{\varepsilon} f_z'\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon*}\theta\right\rangle$$
$$+\varepsilon f_z''\left\langle W_z^\varepsilon, \mathcal{A}^\varepsilon\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle,$$

where $\mathcal{A}^\varepsilon\theta$ denotes

$$\mathcal{A}^\varepsilon\theta = -\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta - \frac{1}{\varepsilon}\mathcal{L}_z^{\varepsilon*}\theta$$

cf. (50). Hence

$$\begin{aligned}\mathcal{A}^\varepsilon\left[f_z + f_{1,z}^\varepsilon\right] &= f_z'\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + f_z'\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle + f_z''\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon*}\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle \\ &\quad + \varepsilon\left[f_z'\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle + f_z''\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle\right] \\ &= A_1^\varepsilon(z) + A_2^\varepsilon(z) + A_1^\varepsilon(z) + R_1^\varepsilon(z),\end{aligned}$$

where $A_2^\varepsilon(z)$ and $A_3^\varepsilon(z)$ are the coupling terms.

Proposition 2.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_1^\varepsilon(z)| = 0.$$

Proof. By Lemma 2 we have

$$\begin{aligned}|R_1^\varepsilon| &\leqslant \varepsilon\|f''\|_\infty\|W_0\|_2^2\left[\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\|_2\|\tilde{\mathcal{L}}_z^{\varepsilon*}\theta\|_2 + \|\mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^{\varepsilon*}\theta)\|_2\right] \\ &= O\left(\eta^{-1}(|\min(\rho,\gamma^{-1})|^{1-H} + \rho^{2-H})\right),\end{aligned} \tag{81}$$

which is of the following order of magnitude

$$\begin{cases} \varepsilon & \text{if } \eta, \rho \text{ held fixed,} \\ \varepsilon\rho^{2-H} & \text{if } \eta, \gamma \text{ held fixed,} \\ \varepsilon\eta^{-1} & \text{if } \rho \text{ is held fixed,} \\ \varepsilon\eta^{-1}\rho^{2-H} & \text{if } \gamma \text{ held fixed,} \\ \varepsilon(|\min(\rho,\gamma^{-1})|^{1-H} + \rho^{2-H}) & \text{if } \eta \text{ held fixed} \end{cases} \tag{82}$$

and vanishes in the respective regimes. ▮

We introduce the next perturbations $f^{\varepsilon}_{2,z}$, $f^{\varepsilon}_{3,z}$. Let

$$A_2^{(1)}(\phi) \equiv \int \phi(\mathbf{x},\mathbf{p})\mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\phi(\mathbf{y},\mathbf{q})\,d\mathbf{x}\,d\mathbf{p}\,d\mathbf{y}\,d\mathbf{q}, \tag{83}$$

$$A_1^{(1)}(\phi) \equiv \int \mathcal{Q}_1'\theta(\mathbf{x},\mathbf{p})\phi(\mathbf{x},\mathbf{p})\,d\mathbf{x}\,d\mathbf{p}, \tag{84}$$

where

$$\mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) = \mathbb{E}\left[\mathcal{L}_z^{\varepsilon*}\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^{\varepsilon*}\theta(\mathbf{y},\mathbf{q})\right] \tag{85}$$

and

$$\mathcal{Q}_1'\theta(\mathbf{x},\mathbf{p}) = \mathbb{E}\left[\mathcal{L}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta(\mathbf{x},\mathbf{p})\right],$$

where the operator $\tilde{\mathcal{L}}_z^{\varepsilon*}$ is defined as in (61). Note that $\mathcal{Q}_1\theta$ and $\mathcal{Q}_1'\theta$ are $O(1)$ terms because of (59).

Clearly, we have

$$A_2^{(1)}(\phi) = \mathbb{E}\left[\langle\phi, \mathcal{L}_z^{\varepsilon*}\theta\rangle\left\langle\phi, \tilde{\mathcal{L}}_z^{\varepsilon*}\theta\right\rangle\right]. \tag{86}$$

Define

$$f^{\varepsilon}_{2,z} \equiv f_z'' \int_z^{\infty} \mathbb{E}_z^{\varepsilon}\left[\langle W_z^{\varepsilon}, \mathcal{L}_s^{\varepsilon*}\theta\rangle\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_s^{\varepsilon*}\theta\right\rangle - A_2^{(1)}(W_z^{\varepsilon})\right] ds$$

$$f^{\varepsilon}_{3,z} \equiv f_z' \int_z^{\infty} \mathbb{E}_z^{\varepsilon}\left[\left\langle W_z^{\varepsilon}, \mathcal{L}_s^{\varepsilon*}\tilde{\mathcal{L}}_s^{\varepsilon*}\theta\right\rangle - A_3^{(1)}(W_z^{\varepsilon})\right] ds.$$

Let

$$\mathcal{Q}_2(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) \equiv \mathbb{E}\left[\tilde{\mathcal{L}}_z^{\varepsilon*}\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^{\varepsilon*}\theta(\mathbf{y},\mathbf{q})\right]$$

and

$$\mathcal{Q}_2'\theta(\mathbf{x},\mathbf{p}) = \mathbb{E}\left[\tilde{\mathcal{L}}_z^{\varepsilon*}\tilde{\mathcal{L}}_z^{\varepsilon*}\theta(\mathbf{x},\mathbf{p})\right].$$

Let

$$A_2^{(2)}(\phi) \equiv \int \phi(\mathbf{x},\mathbf{p})\mathcal{Q}_2(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\phi(\mathbf{y},\mathbf{q})\, d\mathbf{x}\, d\mathbf{p}\, d\mathbf{y}\, d\mathbf{q}, \tag{87}$$

$$A_1^{(2)}(\phi) \equiv \int \mathcal{Q}_2'\theta(\mathbf{x},\mathbf{p})\phi(\mathbf{x},\mathbf{p})\, d\mathbf{x}\, d\mathbf{p}, \tag{88}$$

we then have

$$f_{2,z}^{\varepsilon} = \frac{\varepsilon^2}{2} f_z''\left[\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle^2 - A_2^{(2)}(W_z^{\varepsilon})\right], \tag{89}$$

$$f_{3,z}^{\varepsilon} = \frac{\varepsilon^2}{2} f_z'\left[\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle - A_3^{(2)}(W_z^{\varepsilon})\right]. \tag{90}$$

Using Assumption (66) and the Cauchy–Schwartz inequality one can easily prove the following.

Proposition 3.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_{j,z}^{\varepsilon}| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}|f_{j,z}^{\varepsilon}| = 0, \quad j = 2, 3.$$

We have

$$\mathcal{A}^{\varepsilon} f_{2,z}^{\varepsilon} = f_z''\left[-\left\langle W_z^{\varepsilon}, \mathcal{L}_z^{\varepsilon *}\theta\right\rangle\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle + A_2^{(1)}(W_z^{\varepsilon})\right] + R_2^{\varepsilon}(z),$$

$$\mathcal{A}^{\varepsilon} f_{3,z}^{\varepsilon} = f_z'\left[-\left\langle W_z^{\varepsilon}, \mathcal{L}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle + A_3^{(1)}(W_z^{\varepsilon})\right] + R_3^{\varepsilon}(z)$$

with

$$\begin{aligned} R_2^{\varepsilon}(z) = {} & \frac{\varepsilon^2}{2} f_z'''\left[\left\langle W_z^{\varepsilon}, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^{\varepsilon}, \mathcal{L}_z^{\varepsilon *}\theta\right\rangle\right]\left[\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle^2 - A_2^{(2)}(W_z^{\varepsilon})\right] \\ & + \varepsilon^2 f_z''\left\langle W_z^{\varepsilon}, \tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle\left[\left\langle W_z^{\varepsilon}, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^{\varepsilon *}\theta)\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^{\varepsilon}, \mathcal{L}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle\right] \\ & - \varepsilon^2 f_z'\left[\left\langle W_z^{\varepsilon}, \mathbf{p}\cdot\nabla_{\mathbf{x}}(G_\theta^{(2)} W_z^{\varepsilon})\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^{\varepsilon}, \mathcal{L}_z^{\varepsilon *} G_\theta^{(2)} W_z^{\varepsilon}\right\rangle\right], \end{aligned} \tag{91}$$

where $G_\theta^{(2)}$ denotes the operator

$$G_\theta^{(2)}\phi \equiv \int \mathcal{Q}_2(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\phi(\mathbf{y},\mathbf{q})\, d\mathbf{y}\, d\mathbf{q}.$$

Similarly

$$R_3^\varepsilon(z) = \varepsilon^2 f_z' \left[\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta)\right\rangle + \frac{\tilde{k}}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle\right]$$
$$+\frac{\varepsilon^2}{2} f_z'' \left[\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon *}\theta\right\rangle\right]\left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\rangle - A_1^{(2)}(W_z^\varepsilon)\right]$$
$$-\varepsilon^2 f_z' \left[\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\theta)\right\rangle + \frac{1}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon *}\mathcal{Q}_2'\theta\right\rangle\right]. \tag{92}$$

Proposition 4.

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_2^\varepsilon(z)| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_3^\varepsilon(z)| = 0.$$

Proof. Part of the argument is analogous to that given for Proposition 3. The additional estimates that we need to consider are the following. In R_2^ε (91):

$$\sup_{z<z_0}\varepsilon^2\,\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(G_\theta^{(2)}W_z^\varepsilon)\right\rangle\right|$$
$$\leqslant c\varepsilon^2\gamma^{-2}\|\theta\|_2\|W_0\|_2^2\left\|\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{x}}\mathcal{F}_2^{-1}\theta\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2\right\|_2$$
$$\leqslant c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-1}\left\|[\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{x}}\theta](\mathbf{x},\mathbf{y})\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right\|_2$$
$$+c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-2}\left\|[\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\theta](\mathbf{x},\mathbf{y})\cdot\nabla_{\mathbf{y}}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right\|_2$$
$$\leqslant c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-1}\sup_{|\mathbf{y}|\leqslant L}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y}) + c\|\theta\|_2\|W_0\|_2^2\varepsilon^2\gamma^{-2}$$
$$\sup_{|\mathbf{y}|\leqslant L}\left|\nabla_{\mathbf{y}}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right|$$
$$\leqslant O\left(\varepsilon^2\eta^{-2}\gamma|\min(\rho,\gamma^{-1})|^{2-2H} + \varepsilon^2\eta^{-2}\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}\right)$$

by Lemma 2, where L is the radius of the ball containing the support of θ. Further delineation yields the following order-of-magnitude estimates

$$\begin{cases} \varepsilon^2 & \text{if } \eta,\rho \text{ held fixed,}\\ \varepsilon^2\rho^{1-H} & \text{if } \eta,\gamma \text{ held fixed,}\\ \varepsilon^2\eta^{-2}\rho^{1-H} & \text{if } \gamma \text{ held fixed,}\\ \varepsilon^2\eta^{-2} & \text{if } \rho \text{ held fixed,}\\ \varepsilon^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H} & \text{if } \eta \text{ held fixed.} \end{cases}$$

Consider the next term:

$$
\begin{aligned}
&\sup_{z<z_0} \varepsilon \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon *} G_\theta^{(2)} W_z^\varepsilon \right\rangle\right| \\
&\leqslant c\varepsilon^2 \gamma^{-3} \|W_0\|_2 \mathbb{E}\left\{ \left\| \delta_{21} V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \right.\right. \\
&\quad \left.\left. \times \int \mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x}',\mathbf{y}')\right] \mathcal{F}_2^{-1}\theta(\mathbf{x}',\mathbf{y}')\mathcal{F}_2^{-1} W_z^\varepsilon(\mathbf{x}',\mathbf{y}') d\mathbf{x}' d\mathbf{y}' \right\|_2 \right\} \\
&\leqslant c\varepsilon^2 \gamma^{-3} \|W_0\|_2 \mathbb{E}\left\{ \left\| \delta_{21} V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2 \right.\right. \\
&\qquad\qquad \left.\left. \times \int \left|\mathcal{F}_2^{-1}\theta(\mathbf{x}',\mathbf{y}')\mathcal{F}_2^{-1} W_z^\varepsilon(\mathbf{x}',\mathbf{y}')\right| d\mathbf{x}' d\mathbf{y}' \right\|_2 \right\} \\
&\leqslant c\varepsilon^2 \gamma^{-3} \|\theta\|_2 \|W_0\|_2^2 \mathbb{E}\left\| \delta_{21} V_z^\varepsilon(\mathbf{x},\mathbf{y}) \mathcal{F}_2^{-1}\theta \mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2 \right\|_2 \\
&\leqslant O\left(\varepsilon^2 \eta^{-2} |\min(\rho,\gamma^{-1})|^{3-3H}\right)
\end{aligned}
$$

by Corollary 3.

In R_3^ε (92):

$$
\begin{aligned}
\sup_{z<z_0} \varepsilon \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta \right\rangle\right| &\leqslant \varepsilon \|W_0\|_2 \sup_{z<z_0} \sqrt{\mathbb{E}\left\|\mathcal{L}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\tilde{\mathcal{L}}_z^{\varepsilon *}\theta\right\|_2^2} \\
&= O\left(\varepsilon\gamma^{-3} \sup_{|\mathbf{y}|\leqslant L} \mathbb{E}\left|\delta_{21}\tilde{V}_z^\varepsilon\right|^2(\mathbf{y}) \mathbb{E}^{1/2}\left|\delta_{21}V_z^\varepsilon\right|^2(\mathbf{y})\right) \\
&= O\left(\varepsilon\eta^{-2}|\min(\rho,\gamma^{-1})|^{3-3H}\right),
\end{aligned}
$$

by (72) and Lemma 2. The preceding two terms can be estimated from above by the following order of magnitude:

$$
\begin{cases}
\varepsilon & \text{if } \rho \text{ and } \eta \text{ held fixed,} \\
\varepsilon & \text{if } \gamma \text{ and } \eta \text{ held fixed,} \\
\varepsilon\eta^{-2} & \text{if } \gamma \text{ or } \rho \text{ held fixed,} \\
\varepsilon|\min(\rho,\gamma^{-1})|^{3-3H} & \text{if } \eta \text{ held fixed,}
\end{cases}
$$

$$
\begin{aligned}
\varepsilon^2 \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\theta)\right\rangle\right| &\leqslant \varepsilon^2 \sqrt{\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\theta)\right\rangle\right|^2} \\
&\leqslant c\varepsilon^2\gamma^{-2}\|W_0\|_2 \left\|\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{x}}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2 \mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right\|_2
\end{aligned}
$$

$$= O\left(\varepsilon^2\gamma^{-2}\mathbb{E}_{|\mathbf{y}|\leqslant L}\left|\nabla_{\mathbf{y}}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right|\right)$$
$$= O\left(\varepsilon^2\eta^{-2}\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}\right), \tag{93}$$

which in the various regimes has the following order of magnitude

$$\begin{cases} \varepsilon^2 & \text{if } \rho \text{ and } \eta \text{ held fixed,} \\ \varepsilon^2\rho^{1-H} & \text{if } \gamma \text{ and } \eta \text{ held fixed,} \\ \varepsilon^2\eta^{-2}\rho^{1-H} & \text{if } \gamma \text{ held fixed,} \\ \varepsilon^2\eta^{-2} & \text{if } \rho \text{ held fixed,} \\ \varepsilon^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H} & \text{if } \eta \text{ held fixed,} \end{cases}$$

$$\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon*}\mathcal{Q}_2'\theta\right\rangle\right|$$
$$\leqslant \varepsilon\sqrt{\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^{\varepsilon*}\mathcal{Q}_2'\theta\right\rangle\right|^2}$$
$$\leqslant c\varepsilon^2\gamma^{-3}\|W_0\|_2\mathbb{E}\left\|\delta_{21}V_z^\varepsilon(\mathbf{x},\mathbf{y})\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y})\right\|_2$$
$$= O\left(\varepsilon^2\gamma^{-3}\sup_{|\mathbf{y}|\leqslant L}\mathbb{E}\left|\delta_{21}\tilde{V}_z^\varepsilon\right|^2(\mathbf{y})\mathbb{E}^{1/2}\left|\delta_{21}V_z^\varepsilon\right|^2(\mathbf{y})\right)$$
$$= O\left(\varepsilon^2\eta^{-2}|\min(\rho,\gamma^{-1})|^{3-3H}\right) \tag{94}$$

by Lemma 2. ∎

Consider the test function $f_z^\varepsilon = f_z + f_{1,z}^\varepsilon + f_{2,z}^\varepsilon + f_{3,z}^\varepsilon$. We have

$$\mathcal{A}^\varepsilon f_z^\varepsilon = f_z'\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + f_z'' A_2^{(1)}(W_z^\varepsilon) + f' A_1^{(1)}(W_z^\varepsilon) + R_2^\varepsilon(z) + R_3^\varepsilon(z) + R_1^\varepsilon(z). \tag{95}$$

Set

$$R^\varepsilon(z) = R_1^\varepsilon(z) + R_2^\varepsilon(z) + R_3^\varepsilon(z). \tag{96}$$

It follows from Propositions 2 and 4 that

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R^\varepsilon(z)| = 0.$$

For the tightness it remains to show

Proposition 5. $\{\mathcal{A}^\varepsilon f_z^\varepsilon\}$ are uniformly integrable.

Proof. Indeed, each term in the expression (95) is uniformly integrable. We only need to be concerned with terms in $R^\varepsilon(z)$ since other terms are obviously uniformly integrable because W_z^ε is uniformly bounded in the square norm. But since the previous estimates establish the uniform boundedness of the second moments of the corresponding terms, the uniform integrability of the terms follow. ∎

4.2. Identification of the Limit

Our strategy is to show directly that in passing to the weak limit the limiting process solves the martingale problem formulated in Section 3.1. The uniqueness of the martingale solution mentioned in Section 2.2 then identifies the limiting process as the unique $L^2(\mathbb{R}^{2d})$-valued solution to the initial value problem of the stochastic integral-PDE of the white-noise model.

Recall that for any C^2-function f

$$\begin{aligned} M_z^\varepsilon(\theta) &= f_z^\varepsilon - \int_0^z \mathcal{A}^\varepsilon f_s^\varepsilon\, ds \\ &= f_z + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z) - \int_0^z f_z' \left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle ds \\ &\quad - \int_0^z \left[f_s'' A_2^{(1)}(W_s^\varepsilon) + f_s' A_1^{(1)}(W_s^\varepsilon)\right] ds - \int_0^z R^\varepsilon(s)\, ds \end{aligned} \tag{97}$$

is a martingale. The martingale property implies that for any finite sequence $0<z_1<z_2<z_3<\cdots<z_n\leqslant z$, C^2-function f and bounded continuous function h with compact support, we have

$$\begin{aligned} \mathbb{E}\left\{h\left(\left\langle W_{z_1}^\varepsilon,\theta\right\rangle, \left\langle W_{z_2}^\varepsilon,\theta\right\rangle,\ldots,\left\langle W_{z_n}^\varepsilon,\theta\right\rangle\right)\left[M_{z+s}^\varepsilon(\theta)-M_z^\varepsilon(\theta)\right]\right\} = 0, \\ \forall s>0, \quad z_1\leqslant z_2\leqslant\cdots\leqslant z_n\leqslant z. \end{aligned} \tag{98}$$

Let

$$\bar{\mathcal{A}} f_z \equiv f_s'\left[\langle W_z, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle + \bar{A}_1(W_z)\right] + f_z''\bar{A}_2(W_z),$$

where

$$\bar{A}_2(\phi) = \lim_{\rho\to\infty} A_2^{(1)}(\phi) = \int \overline{Q}(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\phi(\mathbf{x},\mathbf{p})\phi(\mathbf{y},\mathbf{q})d\mathbf{x}\,d\mathbf{p}\,d\mathbf{y}\,d\mathbf{q}, \tag{99}$$

$$\bar{A}_1(\phi) = \lim_{\rho\to\infty} A_1^{(1)}(\theta) = \int \overline{Q}_0(\theta)(\mathbf{x},\mathbf{p})\phi(\mathbf{x},\mathbf{p})d\mathbf{x}\,d\mathbf{p}, \tag{100}$$

where $\overline{Q}(\theta\otimes\theta)$ and $\overline{Q}_0(\theta)$ are given by (26) and (25), respectively. Equations (99) and (100) are obtained by an explicit and tedious calculation.

For $\rho\to\infty, \gamma\to 0$ as $\varepsilon\to 0$ the limits in (99) are not well-defined unless $H\in(0,1/2)$ in the worst case scenario allowed by (6). Likewise, the convergence does not hold for $H\in[1/2,1)$ when $\eta\to 0$ in the worst case scenario allowed by (6).

In view of the results of Propositions 1–4 we see that f_z^ε and $\mathcal{A}^\varepsilon f_z^\varepsilon$ in (97) can be replaced by f_z and $\bar{\mathcal{A}}f_z$, respectively, modulo an error that vanishes as $\varepsilon\to 0$. With this and the tightness of $\{W_z^\varepsilon\}$ we can pass to the limit $\varepsilon\to 0$ in (98).We see that the limiting process satisfies the martingale property that

$$\mathbb{E}\left\{h\left(\left\langle W_{z_1},\theta\right\rangle,\left\langle W_{z_2},\theta\right\rangle,\dots,\left\langle W_{z_n},\theta\right\rangle\right)\left[M_{z+s}(\theta)-M_z(\theta)\right]\right\}=0,\quad \forall s>0,$$

where

$$M_z(\theta)=f_z-\int_0^z \bar{\mathcal{A}}f_s\,ds. \tag{101}$$

Then it follows that

$$\mathbb{E}\left[M_{z+s}(\theta)-M_z(\theta)|W_u,u\leqslant z\right]=0,\quad \forall z,s>0,$$

which proves that $M_z(\theta)$ is a martingale.

Note that $\left\langle W_z^\varepsilon,\theta\right\rangle$ is uniformly bounded:

$$\left|\left\langle W_z^\varepsilon,\theta\right\rangle\right|\leqslant\|W_0\|_2\|\theta\|_2$$

so we have the convergence of the second moment

$$\lim_{\varepsilon\to 0}\mathbb{E}\left\{\left\langle W_z^\varepsilon,\theta\right\rangle^2\right\}=\mathbb{E}\left\{\langle W_z,\theta\rangle^2\right\}.$$

Using $f(r)=r$ and r^2 in (101) we see that

$$M_z^{(1)}(\theta)=\langle W_z,\theta\rangle-\int_0^z\left[\langle W_s,\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle-\bar{A}_3(W_s)\right]ds$$

is a martingale with the quadratic variation

$$\left[M^{(1)}(\theta),M^{(1)}(\theta)\right]_z=\int_0^z\bar{A}_2(W_s)\,ds=\int_0^z\left\langle W_s,\overline{\mathcal{K}}_\theta W_s\right\rangle ds,$$

where $\overline{\mathcal{K}}_\theta$ is defined as in (25).

5. APPENDIX A. PROOF OF LEMMA 2

(i) Estimation of $\sup_{\substack{|z|\leqslant z_0 \\ |\mathbf{y}|\leqslant L}} \mathbb{E}\left[\left(\delta_{21} V_z^\varepsilon\right)^2\right](\mathbf{y})$: We have that for $\gamma\rho\leqslant 1$

$$\begin{aligned}
&\sup_{|z|\leqslant z_0} \mathbb{E}\left[\left(\delta_{21} V_z^\varepsilon(\mathbf{x},\mathbf{y})\right)^2\right] \\
&\quad\leqslant \int |\gamma \mathbf{y}\cdot\mathbf{k}|^2\, \Phi_{(\eta,\rho)}(\xi,\mathbf{k}) d\xi d\mathbf{k} \\
&\quad\leqslant c_0\gamma^2|\mathbf{y}|^2 \int_{|\xi|\leqslant\rho}\int_{|\mathbf{k}|\leqslant\rho} (\eta^2+|\mathbf{k}|^2+|\xi|^2)^{-H-(d+1)/2}|\mathbf{k}|^{d+1} d|\mathbf{k}|d\xi \\
&\quad\leqslant c_3\gamma^2|\mathbf{y}|^2\left(\eta^{2-2H}+\rho^{2-2H}\right).
\end{aligned}$$

For $\rho\gamma\geqslant 1$ we divide the domain of integration into $I_0=\{|\mathbf{k}|\leqslant\gamma^{-1}\}$ and $I_1=\{|\mathbf{k}|\geqslant\gamma^{-1}\}$ and estimate their contributions separately. For I_0 the upper bound is similar to the above, namely, we have

$$\int_{I_0} 4|\sin(\gamma\mathbf{y}\cdot\mathbf{k}/2)|^2\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} \leqslant c_4\gamma^2|\mathbf{y}|^2\left(\eta^{2-2H}+\gamma^{-2+2H}\right).$$

For I_1 we have instead that

$$\begin{aligned}
\int_{I_1} 4|\sin(\gamma\mathbf{y}\cdot\mathbf{k}/2)|^2\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} &\leqslant 4\int_{I_1}\Phi_{(\eta,\rho)}(\xi,\mathbf{k})d\xi d\mathbf{k} \\
&\leqslant c_7\left(\gamma^{2H}+\rho^{-2H}\right).
\end{aligned}$$

Put together, the upper bound becomes

$$\sup_{\substack{|z|\leqslant z_0 \\ |\mathbf{x}|,|\mathbf{y}|\leqslant L}} \mathbb{E}\left[\left(\delta_{21} V_z^\varepsilon(\mathbf{x},\mathbf{y})\right)^2\right] \leqslant c_8\gamma^2\left|\min(\gamma^{-1},\rho)\right|^{2-2H}, \quad \gamma,\eta\leqslant 1\leqslant\rho.$$

(ii) Estimation of $\sup_{|z|\leqslant z_0}\mathbb{E}\left[\tilde{V}_z^\varepsilon(\mathbf{x})\right]^2$: It follows from the argument for Corollary 1 and Assumption 2 that

$$\begin{aligned}
\mathbb{E}\left[\tilde{V}_z^\varepsilon(\mathbf{x})\right]^2 &\leqslant \left(\int_0^\infty r_{\eta,\rho}(t)dt\right)^2 \mathbb{E}[V_z^\varepsilon]^2 \\
&\leqslant c\eta^{-2}\eta^{-2H}.
\end{aligned}$$

(iii) Estimation of $\sup_{\substack{|z|\leqslant z_0\\ |\mathbf{y}|\leqslant L}} \mathbb{E}\left[\left(\delta_{21}\tilde{V}_z^\varepsilon\right)^2\right](\mathbf{y})$: First note that the correlation coefficient for $\delta_{21}\tilde{V}_z^\varepsilon$ is bounded from above by $cr_{\eta,\rho}(t)$ for some constant $c>0$. Then we have as in (i) and (ii) that

$$\begin{aligned}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon(\mathbf{x})\right]^2 &\leqslant c_1\left(\int_0^\infty r_{\eta,\rho}(t)dt\right)^2\mathbb{E}[\delta_{21}V_z^\varepsilon]^2\\ &\leqslant c_2\eta^{-2}\gamma^2\left|\min(\gamma^{-1},\rho)\right|^{2-2H}.\end{aligned}$$

(iv) Estimation of $\sup_{\substack{|z|\leqslant z_0\\ |\mathbf{y}|\leqslant L}}\left|\nabla_\mathbf{y}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right|$: By the Cauchy–Schwartz inequality and the preceding calculation we have

$$\begin{aligned}&\sup_{\substack{|z|\leqslant z_0\\ |\mathbf{y}|\leqslant L}}\left|\nabla_\mathbf{y}\mathbb{E}\left[\delta_{21}\tilde{V}_z^\varepsilon\right]^2(\mathbf{y})\right|\\ &\leqslant c_1\sqrt{\gamma^2\mathbb{E}\left[\nabla_\mathbf{x}\tilde{V}^\varepsilon(\mathbf{x}+\gamma\mathbf{y}/2)+\nabla_\mathbf{x}\tilde{V}^\varepsilon(\mathbf{x}-\gamma\mathbf{y}/2)\right]^2}\sqrt{\mathbb{E}\left[\delta_{21}\tilde{V}^\varepsilon(\mathbf{x},\mathbf{y})\right]^2}\\ &\leqslant c_3\left(\int_0^\infty r_{\eta,\rho}(t)dt\right)^2\gamma\mathbb{E}^{1/2}[\nabla_\mathbf{x}V^\varepsilon]^2\,\mathbb{E}^{1/2}\left[\delta_{21}V_z^\varepsilon(\mathbf{x},\mathbf{y})\right]^2\\ &\leqslant c_4\eta^{-2}\gamma^2\rho^{1-H}|\min(\rho,\gamma^{-1})|^{1-H}\end{aligned}$$

(v) Estimation of $\sup_{|z|\leqslant z_0}\mathbb{E}\|\mathbf{p}\cdot\nabla_\mathbf{x}(\tilde{\mathcal{L}}_z^{\varepsilon*}\theta)\|_2^2$: A similar line of reasoning and a straightforward spectral calculation yield that

$$\begin{aligned}\mathbb{E}\|\mathbf{p}\cdot\nabla_\mathbf{x}(\tilde{\mathcal{L}}_z^{\varepsilon*}\theta)\|_2^2 &= \mathbb{E}\|\nabla_\mathbf{y}\cdot\nabla_\mathbf{x}\gamma^{-1}\delta_{21}\tilde{V}_z^\varepsilon\mathcal{F}_2^{-1}\theta\|_2^2\\ &\leqslant c_1\mathbb{E}\|\nabla_\mathbf{x}^2\tilde{V}_z^\varepsilon\mathcal{F}_2^{-1}\theta\|_2^2\\ &\leqslant c_2\eta^{-2}\mathbb{E}\left[\nabla_\mathbf{x}^2V_z^\varepsilon\right]^2\\ &\leqslant c_3\eta^{-2}\rho^{4-2H}.\end{aligned}$$

6. APPENDIX B. EXACT AND ASYMPTOTIC SOLUTIONS FOR THE GEOMETRICAL OPTICS

In this section we construct the Green function for the geometrical optics equation

$$\frac{\partial \bar{W}_z}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}_z = -\tilde{k}\left(-i\nabla_{\mathbf{p}} + \frac{\beta}{2\tilde{k}}\mathbf{x}\right)\cdot\mathbf{D}(0)\cdot\left(-i\nabla_{\mathbf{p}} + \frac{\beta}{2\tilde{k}}\mathbf{x}\right)\bar{W}_z(\mathbf{x},\mathbf{p}) \\ -\beta^2 D_0(0)\bar{W}_z(\mathbf{x},\mathbf{p}).$$

For simplicity of notation, let us assume isotropy of the medium, namely $\Phi(0,\mathbf{p}) = \Phi(0,|\mathbf{p}|)$ and hence $\mathbf{D}(0) = D(0)$, a scalar. Taking the inverse Fourier transform $\mathcal{F}_2^{-1}$ in $\mathbf{p}$ we obtain

$$\frac{\partial}{\partial z}\hat{W} = i\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}\hat{W} - \frac{D(0)}{\tilde{k}}\left|-\tilde{k}\mathbf{y} + \frac{\beta}{2}\mathbf{x}\right|^2\hat{W} - \beta^2 D_0(0)\hat{W}. \tag{B.1}$$

Introducing the new variables

$$\mathbf{y}_1 = \tilde{k}\mathbf{y} + \frac{\beta}{2}\mathbf{x}, \tag{B.2}$$

$$\mathbf{y}_2 = \tilde{k}\mathbf{y} - \frac{\beta}{2}\mathbf{x}, \tag{B.3}$$

we rewrite the above equation as

$$\frac{\partial}{\partial z}\hat{W} = \frac{i\tilde{k}\beta}{2}\left(\nabla_1^2 - \nabla_2^2\right)\hat{W} - \frac{D(0)}{\tilde{k}}|\mathbf{y}_2|^2\hat{W} - \beta^2 D_0(0)\hat{W}, \tag{B.4}$$

where ∇_1, ∇_2 are the gradients with respect to $\mathbf{y}_1, \mathbf{y}_2$, respectively.

Consider the function

$$\widetilde{W}(z,\mathbf{p}_1,\mathbf{y}_2) = e^{\beta^2 D_0(0)z} e^{i\tilde{k}\beta|\mathbf{p}_1|^2 z/2}\frac{1}{(2\pi)^d}\int \hat{W}\left(z, \frac{\mathbf{y}_1-\mathbf{y}_2}{\beta}, \frac{\mathbf{y}_1+\mathbf{y}_2}{2\tilde{k}}\right)e^{-i\mathbf{y}_1\cdot\mathbf{p}_1}\,d\mathbf{y}_1, \tag{B.5}$$

which satisfies the equation

$$\frac{\partial}{\partial z}\widetilde{W} = -\frac{i\tilde{k}\beta}{2}\nabla_2^2\widetilde{W} - \frac{D(0)}{\tilde{k}}|\mathbf{y}_2|^2\widetilde{W}. \tag{B.6}$$

Equation (B.6) is just the Schrödinger equation with an imaginary, quadratic potential and can be solved by separating the variables $\mathbf{y}_2 = (y_1, y_2, \dots, y_d)$, solving the one-dimensional version of the equation:

$$\frac{\partial}{\partial z}\widetilde{W}_j = -\frac{i\tilde{k}\beta}{2}\frac{\partial^2}{\partial y_j^2}\widetilde{W}_j - \frac{D(0)}{\tilde{k}}y_j^2\widetilde{W}_j, \quad j = 1, 2, \dots, d \tag{B.7}$$

and forming tensor product $\prod_{j=1}^d W_j(y_j)$.

We begin by searching for solutions of the Gaussian form

$$\widetilde{W}_j = e^{-A(z)-B(z)|y_j-C(z)|^2}, \tag{B.8}$$

where A, B, C are complex-valued functions of z, parametrized by $\mathbf{p}_1$. Substituting (B.8) into Eq. (B.6) and comparing the coefficients we obtain the ODEs governing A, B, C:

$$B' = \frac{D(0)}{\tilde{k}} + i2\tilde{k}\beta B^2, \tag{B.9}$$

$$C' = -\frac{D(0)}{\tilde{k}B}C, \tag{B.10}$$

$$A' = \frac{D(0)C^2}{\tilde{k}} - i\tilde{k}\beta B, \tag{B.11}$$

which can be solved in the order of B, C, A and yield

$$\begin{aligned} B(z) &= \frac{1}{\tilde{k}(1-i)}\sqrt{\frac{D(0)}{\beta}}\,\frac{Ke^{2\sqrt{D(0)\beta}(1-i)z}+1}{Ke^{2\sqrt{D(0)\beta}(1-i)z}-1}, \\ C(z) &= C(0)\exp\left[-\frac{D(0)}{\tilde{k}}\int_0^z B(s)^{-1}ds\right], \end{aligned} \tag{B.12}$$

where the constant K is given by

$$K = \frac{1+(1+i)/(2B(0)\tilde{k})\sqrt{\frac{D(0)}{\beta}}}{1-(1+i)/(2B(0)\tilde{k})\sqrt{\frac{D(0)}{\beta}}}.$$

First we look for the Green function in the coordinates $\mathbf{y}_1, \mathbf{y}_2$. To this end, we set $K=1$ corresponding to $B(0)=+\infty$, $j=1,2,\dots,d$, and write

$$A(z)=-i\tilde{k}\beta\int_1^z B(s)ds+\frac{D(0)}{\tilde{k}}\int_0^z C^2(s)ds, \quad z>0$$

with

$$\begin{aligned} B(z) &= \frac{1}{\tilde{k}(1-i)}\sqrt{\frac{D(0)}{\beta}}\frac{e^{2\sqrt{D(0)\beta}(1-i)z}+1}{e^{2\sqrt{D(0)\beta}(1-i)z}-1} \\ &= \frac{-1}{\tilde{k}(1+i)}\sqrt{\frac{D(0)}{\beta}}\cot\left[\sqrt{D(0)\beta}(1+i)z\right]. \end{aligned}$$

Then a straightforward calculation shows that with a suitable normalizing constant c_0 the Green function is given by

$$\begin{aligned} &G(z,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_1',\mathbf{y}_2') \\ &= c_0 e^{-\beta^2 D_0(0)z} e^{id\tilde{k}\beta\int_1^z B(s)ds-\frac{D(0)}{\tilde{k}}\int_\infty^z |\mathbf{C}|^2(s)ds} e^{-B(z)|\mathbf{y}_2-\mathbf{C}(z)|^2} \\ &\quad\times\int e^{-i\tilde{k}\beta|\mathbf{p}_1|^2 z/2} e^{i\mathbf{p}_1\cdot(\mathbf{y}_1-\mathbf{y}_1')}d\mathbf{p}_1 \\ &= c_0\left(\frac{2\pi}{iz\tilde{k}\beta}\right)^d \exp\left[-\beta^2 D_0(0)z\right]\exp\left[id\tilde{k}\beta\int_1^z B(s)ds\right] \\ &\quad\times\exp\left[\frac{-D(0)}{\tilde{k}}\int_0^z \mathbf{C}^2(s)ds\right]\exp\left[i\frac{|\mathbf{y}_1-\mathbf{y}_1'|^2}{2z\tilde{k}\beta}\right]\exp\left[-B(z)|\mathbf{y}_2-\mathbf{C}(z)|^2\right] \\ &= c_0\left(\frac{2\pi}{iz\tilde{k}\beta\sin^{1/2}\left[\sqrt{D(0)\beta}(1+i)z\right]}\right)^d e^{-\beta^2 D_0(0)z}\exp\left[i\frac{|\mathbf{y}_1-\mathbf{y}_1'|^2}{2z\tilde{k}\beta}\right] \\ &\quad\times\exp\left[-\frac{|\mathbf{y}_2'|^2}{(1+i)\tilde{k}}\sqrt{\frac{D(0)}{\beta}}\tan\left(\sqrt{\beta D(0)}(1+i)z\right)\right] \\ &\quad\times\exp\left[\frac{1}{\tilde{k}(1+i)}\sqrt{\frac{D(0)}{\beta}}\cot\left(\sqrt{D(0)\beta}(1+i)z\right)\right. \\ &\quad\left.\times\left|\mathbf{y}_2-\frac{\mathbf{y}_2'}{\cos\left(\sqrt{D(0)\beta}(1+i)z\right)}\right|^2\right], \end{aligned} \tag{B.13}$$

where $\mathbf{C}(z)=(C_j(z))$ is given by the formula (B.13) with the initial data $\mathbf{C}(0)=\mathbf{y}_2'$. The general solution for Eq. (B.1) can then be expressed as

$$\hat{W}(z,\mathbf{x},\mathbf{y}) = (\tilde{k}\beta)^d \int \hat{W}_0(\mathbf{x}',\mathbf{y}') \times G\left(z,\tilde{k}\mathbf{y}+\frac{\beta}{2}\mathbf{x},\tilde{k}\mathbf{y}-\frac{\beta}{2}\mathbf{x},\tilde{k}\mathbf{y}'+\frac{\beta}{2}\mathbf{x}',\tilde{k}\mathbf{y}'-\frac{\beta}{2}\mathbf{x}'\right) d\mathbf{x}'\,d\mathbf{y}'.$$

With the change of coordinates (39) and (40)

$$\mathbf{x}\approx\frac{\sqrt{\tilde{k}}}{2}(\mathbf{x}_1+\mathbf{x}_2),\qquad \mathbf{y}\approx\frac{\sqrt{\tilde{k}}}{\gamma}(\mathbf{x}_1-\mathbf{x}_2)-\frac{\beta}{4\tilde{k}^{1/2}}(\mathbf{x}_1+\mathbf{x}_2), \tag{B.14}$$

we can express the geometrical optics asymptotics $\gamma\ll 1$ of the mutual coherence function as

$$\begin{aligned}
&\Gamma(z,\mathbf{x}_1,\mathbf{x}_2)\\
&\approx\left(\frac{\tilde{k}^2}{\pi\gamma^2\beta\sqrt{z}}\right)^d\left(\frac{(1+i)\sqrt{D(0)\beta}}{\sin\left[\sqrt{D(0)\beta}(1+i)z\right]}\right)^{d/2}\int\exp\left[i\frac{\tilde{k}^2}{2z\beta\gamma^2}\left|\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_1'+\mathbf{x}_2'\right|^2\right]\\
&\times\exp\left[-\frac{1}{1+i}\left|\frac{\tilde{k}}{\gamma}(\mathbf{x}_1'-\mathbf{x}_2')-\frac{\beta}{2}(\mathbf{x}_1'+\mathbf{x}_2')\right|^2\sqrt{\frac{D(0)}{\beta}}\tan\left(\sqrt{\beta D(0)}(1+i)z\right)\right]\\
&\times\exp\left[\frac{1}{1+i}\sqrt{\frac{D(0)}{\beta}}\cot\left(\sqrt{D(0)\beta}(1+i)z\right)\left|\frac{\tilde{k}}{\gamma}(\mathbf{x}_1-\mathbf{x}_2)\right.\right.\\
&\left.\left.-\frac{\beta}{2}(\mathbf{x}_1+\mathbf{x}_2)-\frac{\frac{\tilde{k}}{\gamma}(\mathbf{x}_1'-\mathbf{x}_2')-\frac{\beta}{2}(\mathbf{x}_1'+\mathbf{x}_2')}{\cos\left(\sqrt{D(0)\beta}(1+i)z\right)}\right|^2\right]\\
&\times\Gamma_0(\mathbf{x}_1',\mathbf{x}_2';\tilde{k}_1,\tilde{k}_2)d\mathbf{x}_1'\,d\mathbf{x}_2'\times\exp\left[-\beta^2D_0(0)z\right].
\end{aligned}$$

Next we consider the long distance asymptotics for $z\gg 1$. We note the following asymptotics:

$$B(z)\sim\frac{1+i}{2\tilde{k}}\sqrt{\frac{D(0)}{\beta}}, \tag{B.15}$$

$$C(z)\sim C(0)e^{-\sqrt{\beta D(0)}(1-i)z}, \tag{B.16}$$

$$A(z)\sim\frac{1-i}{2}\sqrt{\beta D(0)}z \tag{B.17}$$

and hence the leading order asymptotics for the Green function

$$G(z,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_1',\mathbf{y}_2') \sim c_0\left(\frac{2\pi}{iz\tilde{k}\beta}\right)^d \exp\left[-\beta^2 D_0(0)z\right]\exp\left[-\frac{1-i}{2}d\sqrt{\beta D(0)}z\right] \\ \times\exp\left[i\frac{|\mathbf{y}_1-\mathbf{y}_1'|^2}{2z\tilde{k}\beta}\right]\exp\left[-\frac{1+i}{2\tilde{k}}\sqrt{\frac{D(0)}{\beta}}\left(|\mathbf{y}_2|^2+|\mathbf{y}_2'|^2\right)\right]. \quad \text{(B.18)}$$

Using the leading asymptotics of (B.15) and (B.17) in the general formula we find the long distance asymptotics for the mutual coherence function Γ

$$\Gamma(z,\mathbf{x}_1,\mathbf{x}_2) \sim \left(\frac{(1+i)\sqrt{D(0)\beta}\tilde{k}^4}{\pi^2\gamma^4\beta^2 z}\right)^{d/2} \exp[-\beta^2 D_0(0)z] \\ \times\exp\left[-\frac{1-i}{2}d\sqrt{\beta D(0)}z\right] \\ \times\exp\left[-\frac{1+i}{2}\sqrt{\frac{D(0)}{\beta}}\left|\frac{\tilde{k}}{\gamma}(\mathbf{x}_1-\mathbf{x}_2)-\frac{\beta}{2}(\mathbf{x}_1+\mathbf{x}_2)\right|^2\right] \\ \times\int\exp\left[-\frac{1+i}{2}\sqrt{\frac{D(0)}{\beta}}\left|\frac{\tilde{k}}{\gamma}(\mathbf{x}_1'-\mathbf{x}_2')-\frac{\beta}{2}(\mathbf{x}_1'+\mathbf{x}_2')\right|^2\right] \\ \times\exp\left[i\frac{\tilde{k}^2}{2\gamma^2\beta z}|\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_1'+\mathbf{x}_2'|^2\right]\Gamma_0(\mathbf{x}_1',\mathbf{x}_2';\tilde{k}_1,\tilde{k}_2)d\mathbf{x}_1'\,d\mathbf{x}_2'$$

One sees from the above expression that the (rescaled) coherent bandwidth β_c is given by

$$\beta_c \sim \frac{1}{D(0)z^2}, \quad \text{(B.19)}$$

which is consistent with the results given in ref. 9, obtained by making the plane-wave assumption.

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