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Boundary Value Problem with Unbounded, Fast Oscillatory Random Flows

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We use Tartar's weak convergence method in conjunction with a variational principle to prove a sharp homogenization theorem for diffusion in steady random flows. The flow has a stationary and square integrable stream matrix. The key of our approach is introducing approximate correctors by means of a saddle point variational principle. We also obtain the two-term asymptotics.

Key Words: Convection; diffusion; homogenization.

Contents.

1. INTRODUCTION

Let $\boldsymbol{v}(\mathbf{x})$, with $\nabla \cdot \boldsymbol{v}(\mathbf{x}) = 0$, be an incompressible velocity field in R^d, $d \geqslant 2$, and let $T(t, \mathbf{x})$ be the temperature distribution in the fluid with the velocity $\boldsymbol{v}(\mathbf{x})$ and some microscopic heat conductivity σ. The temperature satisfies the convection-diffusion equation

$$\frac{\partial T}{\partial t} + \boldsymbol{v}(\mathbf{x}) \cdot \nabla T = \sigma \, \Delta T, \tag{1.1}$$

with $T(0, \mathbf{x}) = T_0(\mathbf{x})$.

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We shall study the long time, large space scale behavior of $T(t, \mathbf{x})$ described by Eq. (1.1) under the influence of both the velocity $\boldsymbol{v}(\mathbf{x})$ and the microscopic conductivity σ. We rescale Eq. (1.1) with the diffusive scaling

$$\mathbf{x} \to \mathbf{x}/\varepsilon, \qquad t \to t/\varepsilon^2 \tag{1.2}$$

and let the initial data varying with the slow variable $\mathbf{x}$. Thus Eq. (1.1) becomes

$$\frac{\partial T_\varepsilon}{\partial t} + \frac{1}{\varepsilon} \boldsymbol{v}(\mathbf{x}/\varepsilon) \cdot \nabla T_\varepsilon = \sigma \Delta T_\varepsilon, \tag{1.3}$$

with $T_\varepsilon(0, \mathbf{x}) = T_0(\mathbf{x})$. This is particularly relevant when the velocity field has a repetitive structure as, for example, when it is a periodic or a stationary random function with zero mean. Under appropriate conditions an overall diffusive behavior with an *effective* diffusion constant is expected. When this happens, Eq. (1.1) is said to *homogenize*.

The sharp condition under which the effective diffusion takes place is best formulated in terms of the stream matrix $\mathbf{H}(\mathbf{x}, \omega)$ which is skew-symmetric and satisfies

$$\nabla \cdot \mathbf{H}(\mathbf{x}, \omega) = \boldsymbol{v}(\mathbf{x}, \omega), \tag{1.4}$$

where ω denotes the randomness of the flows. Such matrix $\mathbf{H}$ always exists, because $\boldsymbol{v}(\mathbf{x}, \omega)$ is incompressible and has mean zero, but may not be *stationary* even though $\boldsymbol{v}(\mathbf{x}, \omega)$ is stationary. This is due to the random nature of the velocity. But if the dimension is bigger than two and the velocity correlation decays sufficiently fast, say, like a power higher than two, then a *square integrable* stationary stream matrix $\mathbf{H}(\mathbf{x}, \omega)$ can be constructed from $\boldsymbol{v}(\mathbf{x}, \omega)$. In two dimension, a random stationary velocity field generally gives rise to logarithmic divergence in the variance of stream matrix. For such a non-stationary stream matrix, non-diffusive long time behavior is to be expected (see, for examples, Avellaneda *et al.* [2], Bouchaud and Georges [4], Fannjiang [7], Fisher *et al.* [9], Koch and Brady [11], Kravtsov *et al.* [12]) so the diffusive scaling (12) is not appropriate.

The L^2-stationarity of stream matrix is the exact condition of homogenization for steady flows in all dimensions. The sharpness of the condition was demonstrated for steady shear layer flows by Avellaneda and Majda [1]. The homogenization theorems for general steady flows under such a general condition was proved by Fannjiang and Papanicolaou [8]. The purpose of the present paper is to establish similar homogenization theorems for a weaker notion of solutions (1.10) using a simpler alternative method.

We assume throughout this paper that the velocity field comes from a square integrable, stationary stream matrix $\mathbf{H}(\mathbf{x}, \omega)$

$$\langle |\mathbf{H}_{i,j}|^2 \rangle < \infty, \qquad \forall i, j \tag{1.5}$$

and (1.4) is meant in the weak sense. In terms of the stream matrix, Eq. (1.3) can be written in divergence form

$$\frac{\partial T_\varepsilon(t, \mathbf{x}, \omega)}{\partial t} = \nabla \cdot [(\sigma \mathbf{I} + \mathbf{H}(\mathbf{x}/\varepsilon, \omega)) \nabla T_\varepsilon(t, \mathbf{x}, \omega)], \tag{1.6}$$

where $\mathbf{I}$ is the identity matrix. One expects that, as ε tends to zero, T_ε tends to the solution $\overline{T}$

$$\frac{\partial \overline{T}(t, \mathbf{x})}{\partial t} = \sum_{i,j=1}^{d} \frac{1}{2} (\sigma^*_{i,j} + \sigma^*_{j,i}) \frac{\partial^2 \overline{T}(t, \mathbf{x})}{\partial x_i \, \partial x_j}, \qquad \overline{T}(0, \mathbf{x}) = T_0(\mathbf{x}) \tag{1.7}$$

in a suitable sense, where $\sigma^*_{i,j}$ is a constant matrix called the *effective diffusivity*.

Since the stream matrix is time independent we work entirely with time independent problems through the Laplace transform of (1.6)

$$\hat{T}_\varepsilon(\lambda, \mathbf{x}, \omega) = \int_0^\infty e^{-\lambda t} T_\varepsilon(t, \mathbf{x}, \omega) \, dt, \qquad \lambda > 0 \tag{1.8}$$

which satisfies

$$-\nabla \cdot [(\mathbf{I} + \mathbf{H}(\mathbf{x}/\varepsilon, \omega)) \nabla \hat{T}_\varepsilon(\lambda, \mathbf{x}, \omega)] + \lambda \hat{T}_\varepsilon(\lambda, \mathbf{x}, \omega) = T_0(\mathbf{x}), \tag{1.9}$$

for $\mathbf{x} \in R^d$. This is a resolvent equation for the evolution Eq. (1.6). The Dirichlet problem has the weak form

$$\int_D (\mathbf{I} + \mathbf{H}(\mathbf{x}/\varepsilon, \omega)) \nabla \hat{T}_\varepsilon(\lambda, \mathbf{x}, \omega) \cdot \nabla \phi(\mathbf{x}, \omega) \, d\mathbf{x} + \lambda \int_D \hat{T}_\varepsilon(\lambda, \mathbf{x}, \omega) \, \phi(\mathbf{x}, \omega)$$

$$= \int_D T_0(\mathbf{x}) \, \phi(\mathbf{x}, \omega) \, d\mathbf{x} \tag{1.10}$$

for every test function $\phi \in \mathscr{C}^1_0(D)$. The space $\mathscr{C}^1_0(D)$ of test functions is natural in view that we are seeking solutions in the space $W^{1,2}_0(D)$ and $\mathbf{H}(\mathbf{x}, \omega)$ is in $L^2(D)$ almost surely. Since $\mathscr{C}^\infty_c(D)$ is dense in $\mathscr{C}^1_0(D)$ one can work entirely with smooth test functions with compact support.

The proof of Fannjiang and Papanicolaou [8] relies on nonlocal variational principles for the resolvent Eq. (1.10). In *bounded domains* where boundary conditions are present, the nonlocality of the variational principles requires subtle construction of cut-off functions to treat boundary behaviors. With this complication, the evaluation of nonlocal functionals in the limit $\varepsilon \to 0$ is a hard calculation. The gain is the well-posedness result in a suitable space which is not obvious at all problems with unbounded coefficients.

The case of *bounded* random coefficients is solved by Papanicolaou and Varadhan [14]. Their approach is based on Tartar's weak convergence method with oscillatory test functions. Tartar's method is desirable in that it avoids the trouble of dealing with boundary behaviors and thus make the passing to the limit $\varepsilon \to 0$ a relatively simple matter. The difficulty in applying this method for unbounded random coefficients is justifying the use of correctors as legitimate test functions.

Tartar's method was reconsidered by Avellaneda and Majda [1] in the case of unbounded random coefficients. They proved the weak convergence

$$\lim_{\varepsilon \to 0} \int_0^t \int_D \langle (T_\varepsilon(s, \mathbf{x}, \omega) - \bar{T}(s, \mathbf{x}))\, \phi(s, \mathbf{x}) \rangle = 0,$$

$$\forall \phi \in L^2((0, \infty), L^2(D)), \quad \forall t > 0 \tag{1.11}$$

for unbounded flows satisfying

$$\langle |\boldsymbol{v}_k(\mathbf{x}, \cdot)|^{\delta + d/2} \rangle + \langle |\mathbf{H}_{i,j}(\mathbf{x}, \cdot)|^p \rangle < \infty, \qquad i, j, k = 1, 2, ..., d, \tag{1.12}$$

for some $\delta > 0$ where $p = 2 + \delta$, if $d = 2$ and $p = d$ for $d \geqslant 3$. Here $\langle \cdot \rangle$ denotes the ensemble average w.r.t. ω.

The condition (1.12) is needed (cf. Avellaneda and Majda [1]) to show that the correctors are legitimate test functions. Moreover, to control the asymptotic behaviors of the correctors as $\varepsilon \to 0$ the ensemble average in (1.11) is taken. The sharp homogenization theorem for square integrable stream matrices can not be obtained this way because the correctors are only known to be $W^{1,2}$, not $\mathscr{C}^1$, functions.

To overcome this drawback one clearly should use *approximate correctors* of better regularity. In the present paper we obtain the suitable approximate correctors by means of a saddle-point variational principle on the probability space of stream matrices. Our objective is to establish homogenization theorems for the weakest solutions of the convection-diffusion equation with the most general stream matrices. We shall accomplish this by Tartar's energy method with the use of approximate correctors.

THEOREM 1.1. *Let the stream matrix* **H** *be stationary and square integrable. Let* $T_0 \in L^2(D)$. *Then*

1. *Equation* (1.10) *admits a weak solutions* $\hat{T}_\varepsilon(\lambda, t, \mathbf{x})$ (1.10) *satisfying the energy estimate* (3.7).

2. *Any weak solutions* $\{\hat{T}_\varepsilon(\lambda, t, \mathbf{x})\}$ *satisfying the energy estimate* (3.7) *converges strongly*

$$\lim_{\varepsilon \to 0} \int_D |\hat{T}_\varepsilon(\lambda, \mathbf{x}, \omega) - \hat{\bar{T}}(\lambda, \mathbf{x})|^2 \, d\mathbf{x} = 0 \tag{1.13}$$

for almost all ω, *where* $\hat{\bar{T}}(\lambda, t, \mathbf{x})$ *is the solution of the resolvent equation for the heat equation* (1.7) *with the effective diffusivity* σ^* *given in* (4.3).

If the assumption of the energy estimate (3.7) is strengthened to that of the energy equality (3.8) then a stronger result holds:

THEOREM 1.2. *Let the stream matrix* **H** *be stationary and square integrable. Let* $T_0(\mathbf{x})$ *be a* $\mathscr{C}^\infty(D)$ *function. Then any weak solutions* T_ε *satisfying the energy equality* (3.8) *have the two-term asymptotics*

$$\lim_{\varepsilon \to 0} \int_D \left| \nabla \hat{T}_\varepsilon(\lambda, \mathbf{x}) - \nabla \hat{\bar{T}}(\lambda, \mathbf{x}) - \sum_i \frac{\partial \hat{\bar{T}}}{\partial x_i}(\mathbf{x}) \, \nabla \chi_i(\mathbf{x}/\varepsilon) \right|^2 = 0 \tag{1.14}$$

for almost all ω. *Here* χ_i, $i = 1, 2, 3, ..., d$ *are the correctors defined in* (4.8)–(4.9).

Besides the simplicity of this approach, the homogenization theorem obtained here is more general since the notion of weak solutions considered in the present paper is weaker than that of Fannjiang and Papanicolaou [8]. For this notion of weak solutions we do not know if solutions are unique. Nevertheless, the solutions have a unique deterministic limit point which is the solution of the effective equation. Also, the solution produced in Fannjiang and Papanicolaou [8] satisfies the energy equality (3.8) and hence the statements of Theorem 1.2.

The approach advocated in the present paper may be generalized to the case of *time dependent* flows for which the evolution equation can not be reduced to a resolvent equation and thus the approach of [8] would not work directly. It is not clear what the sharp homogenization condition is for time dependent flows. We plan to address this problem in a forthcoming paper.

2. NOTATION AND FORMULATION

2.1. *Stationary Stream Matrix*

Let us review the theory of stationary processes in this section.

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathbf{H}(\mathbf{x}, \omega)$ be a *strictly stationary* random skew-symmetric matrix of $\mathbf{x} \in R^d$ such that each element $\mathbf{H}_{ij}$ is a L^2 function

$$\langle |\mathbf{H}_{ij}(\mathbf{x}, \cdot)|^2 \rangle < \infty, \qquad \forall i, j, \tag{2.1}$$

where $\langle \cdot \rangle$ denotes the average or integral with respect to the measure P. By strict stationarity we mean that the joint distribution of $\mathbf{H}_{ij}(\mathbf{x}_1, \omega)$, $\mathbf{H}_{ij}(\mathbf{x}_2, \omega)$, ..., $\mathbf{H}_{ij}(\mathbf{x}_n, \omega)$ for any points $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n \in R^d$ and that of $\mathbf{H}_{ij}(\mathbf{x}_1 + \boldsymbol{l}, \omega)$, $\mathbf{H}_{ij}(\mathbf{x}_2 + \boldsymbol{l}, \omega)$, ..., $\mathbf{H}_{ij}(\mathbf{x}_n + \boldsymbol{l}, \omega)$ for any $\boldsymbol{l} \in R^d$ is the same, so the averages in (2.1) are independent of $\mathbf{x}$. Without loss of generality (see Doob [5]), we may assume that there is a group of transformations $\tau_{\mathbf{x}}$, $\mathbf{x} \in R^d$ from Ω into Ω that is one to one and preserves the measure P. That is, $\tau_{\mathbf{x}} \tau_{\mathbf{y}} = \tau_{\mathbf{x}+\mathbf{y}}$ and $P(\tau_{\mathbf{x}} A) = P(A)$ for any $A \in \mathscr{F}$. We may also suppose that there is a square integrable (w.r.t. P) matrix function $\tilde{\mathbf{H}}(\omega)$ such that

$$\mathbf{H}(\mathbf{x}, \omega) = \tilde{\mathbf{H}}(\tau_{-\mathbf{x}} \omega), \qquad \mathbf{x} \in R^d, \qquad \omega \in \Omega. \tag{2.2}$$

We assume that the group of transformations $\tau_{\mathbf{x}}$ is ergodic with respect to the probability measure P.

The random stationary divergence free velocity $\boldsymbol{v}$ which we consider in this paper is given by

$$-\boldsymbol{v}(\mathbf{x}, \omega) = \nabla \cdot \mathbf{H}(\mathbf{x}, \omega). \tag{2.3}$$

In two dimension the matrix $\mathbf{H}$ has the form

$$\mathbf{H} = \begin{pmatrix} 0 & -\boldsymbol{h} \\ \boldsymbol{h} & 0 \end{pmatrix} \tag{2.4}$$

where $\boldsymbol{h} = \boldsymbol{h}(\mathbf{x}, \omega)$ is the usual stream function. In three dimensions, $\mathbf{H}$ has the form

$$\mathbf{H} = \begin{pmatrix} 0 & -\boldsymbol{h}_3 & \boldsymbol{h}_2 \\ \boldsymbol{h}_3 & 0 & -\boldsymbol{h}_1 \\ -\boldsymbol{h}_2 & \boldsymbol{h}_1 & 0 \end{pmatrix}. \tag{2.5}$$

where $\boldsymbol{h}(\mathbf{x}, \omega) = (\boldsymbol{h}_1(\mathbf{x}, \omega), \boldsymbol{h}_2(\mathbf{x}, \omega), \boldsymbol{h}_3(\mathbf{x}, \omega))$ is the vector potential of the flow $\boldsymbol{v}$ so that $\nabla \cdot \mathbf{H} = -\nabla \times \boldsymbol{h} = -\boldsymbol{v}$.

We denote the space of square integrable functions on Ω $L^2(\Omega, \mathscr{F}, P)$ by $\mathscr{H}$ which is a Hilbert space with the inner product

$$(\tilde{f}, \tilde{g}) := \langle \tilde{f}\tilde{g} \rangle = \int_{\Omega} P(d\omega)\, \tilde{f}(\omega)\, \tilde{g}(\omega), \qquad \tilde{f}, \tilde{g} \in \mathscr{H}. \tag{2.6}$$

The group of translations $\tau_{\mathbf{x}}$ induces a group of unitary transformations $U_{\mathbf{x}}$ on $\mathscr{H}$ given by

$$(U_{\mathbf{x}} \tilde{f})(\omega) = \tilde{f}(\tau_{-\mathbf{x}}\omega), \qquad \mathbf{x} \in R^d, \quad \omega \in \Omega. \tag{2.7}$$

The unitarity of $U_{\mathbf{x}}$ follows from the measure-preserving of $\tau_{\mathbf{x}}$. In fact, $U_{\mathbf{x}}$ is unitary on all the spaces $L^p(\Omega, \mathscr{F}, P)$, $1 \leqslant p \leqslant \infty$. $\{U_{\mathbf{x}}\}$ have closed densely defined infinitesimal generators $\tilde{\nabla}_i$

$$\tilde{\nabla}_i := \frac{\partial}{\partial x_i} U_{\mathbf{x}} \Big|_{\mathbf{x}=0} \tag{2.8}$$

in each direction $i = 1, 2, ..., d$ with domains $\mathscr{D}_i \subset \mathscr{H}$. The closed subset of $\mathscr{H}$

$$\mathscr{H}^1 = \bigcap_{i=1}^{d} \mathscr{D}_i \tag{2.9}$$

becomes a Hilbert space with the inner product

$$(\tilde{f}, \tilde{g})_1 := \int_{\Omega} P(d\omega)\, \tilde{f}(\omega)\, \tilde{g}(\omega) + \sum_{i=1}^{a} \int_{\Omega} P(d\omega)\, \tilde{\nabla}_i \tilde{f}(\omega)\, \tilde{\nabla}_i \tilde{g}(\omega). \tag{2.10}$$

The ergodic hypothesis on $\tau_{\mathbf{x}}$ implies that the only functions in $\mathscr{H}$ that are invariant under $U_{\mathbf{x}}$ are the constant functions.

Let $H_s(R^d; \mathscr{H})$ be the space of all stationary random processes $f(\mathbf{x}, \omega)$ on R^d, such that $\int_{\Omega} P(d\omega)\, f^2(\mathbf{x}, \omega) = \text{const.} < \infty$. Clearly $H_s(R^d; \mathscr{H})$ is in one-to-one correspondence with $\mathscr{H}$ since it is simply the space of all translates of $\mathscr{H}$, that is, $f(\mathbf{x}, \omega) \in H_s(R^d; \mathscr{H})$ iff $f(\mathbf{x}, \omega) = U_{\mathbf{x}} \tilde{f}(\omega)$, $\tilde{f}(\omega) \in \mathscr{H}$. Similarly, we may identify $\mathscr{H}^1$ with the set of mean square differentiable, stationary processes $H_s^1(R^d; \mathscr{H})$. In particular, if $f \in H_s^1$, then its derivatives are also a stationary processes and

$$\nabla_i f(\mathbf{x}, \omega) = \frac{\partial f(\mathbf{x}, \omega)}{\partial x_i} = U_{\mathbf{x}}(\tilde{\nabla}_i f)(\omega) \tag{2.11}$$

with equality holding $d\mathbf{x} \times P$ almost everywhere. Thus, we have $H_s^1(R^d; \mathscr{H}) = H_s(R^d; \mathscr{H}^1)$.

2.2. *Function Spaces*

Let $\mathscr{V}$ and $\mathscr{V}_\infty$ denote the spaces of square integrable and uniformly bounded vector fields on $(\Omega, \mathscr{F}, P)$ respectively, i.e.,

$$\mathscr{V} := (L^2(\Omega, P))^d \tag{2.12}$$

$$\mathscr{V} := (L^\infty(\Omega, P))^d. \tag{2.13}$$

We define the spaces $\mathscr{V}_g$ of square integrable gradient fields and its zero mean subspace $\check{\mathscr{V}}_g$

$$\mathscr{V}_g := \{\tilde{\mathbf{F}} \in \mathscr{V} \mid \tilde{\nabla} \times \tilde{\mathbf{F}} = 0 \text{ weakly}\} \tag{2.14}$$

$$\check{\mathscr{V}}_g := \{\tilde{\mathbf{F}} \in \mathscr{V}_g \mid \langle \tilde{\mathbf{F}} \rangle = 0\}. \tag{2.15}$$

Complementary to the gradient fields are the space of the curl fields:

$$\mathscr{V}_c := \{\tilde{\mathbf{F}} \in \mathscr{V} \mid \tilde{\nabla} \cdot \tilde{\mathbf{F}} = 0 \text{ weakly}\} \tag{2.16}$$

$$\check{\mathscr{V}}_c := \{\tilde{\mathbf{F}} \in \mathscr{V}_c \mid \langle \tilde{\mathbf{F}} \rangle = 0\}. \tag{2.17}$$

According to the Helmholtz decomposition theorem, the space $\mathscr{V}$ admits the orthogonal decomposition of gradient fields, curl fields and constants

$$\mathscr{V} = \check{\mathscr{V}}_g \oplus \check{\mathscr{V}}_c \oplus R^d, \tag{2.18}$$

where R^d represents the space of constant vector fields.

Next we consider some dense subspaces in $\mathscr{V}_g$ which we will be working with. The first is the space $\mathscr{B}_g$ of bounded gradient fields

$$\mathscr{B}_g := \mathscr{V}_g \cap \mathscr{V}_\infty. \tag{2.19}$$

The second is the space $\mathscr{C}_g$ of bounded, continuous gradient fields

$$\mathscr{C}_g := \{\tilde{\mathbf{F}} \in \mathscr{B}_g \mid \tilde{\mathbf{F}}(\tau_{-\mathbf{x}}\omega) \in (\mathscr{C}(D))^d \text{ a.e. } \omega\}. \tag{2.20}$$

Let us consider the stream matrix $\mathbf{H}$ as a multiplicative transformation from $\mathscr{V}_g$ to $\mathscr{V}$:

$$\tilde{\mathbf{H}}: \tilde{\mathbf{F}} \in \mathscr{V}_g \to \tilde{\mathbf{H}}\tilde{\mathbf{F}}. \tag{2.21}$$

The transformation $\tilde{\mathbf{H}}$ is densely defined since its domain includes $\mathscr{B}_g$, the space of bounded gradient fields.

Consider the orthogonal projection operator $\tilde{\Gamma}$ from $\mathscr{V}$ to $\check{\mathscr{V}}_g$ with the spectral representation given by

$$\begin{cases} \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2} & \text{if} \quad \mathbf{k}\neq 0 \\ 0 & \text{if} \quad \mathbf{k}=0. \end{cases} \tag{2.22}$$

The operator $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}})$ is densely defined on $\mathscr{V}(\mathscr{V}_g)$ because $\tilde{\mathbf{H}}$ is.

We claim

LEMMA 2.1. $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}})$ *is a closable operator on* $\mathscr{V}(\mathscr{V}_g)$.

Proof. We need to check that if a sequence $\tilde{\mathbf{E}}_n\to 0$ in $\mathscr{V}_g$ and $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{E}}_n\to\tilde{\mathbf{G}}$ in $\mathscr{V}_g$ for some $\tilde{\mathbf{G}}$ then $\tilde{\mathbf{G}}=0$. This follows from

$$\langle\tilde{\mathbf{G}}\cdot\tilde{\mathbf{F}}\rangle=\lim_{n\to\infty}\langle\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{E}}_n\cdot\tilde{\mathbf{F}}\rangle=\lim_{n\to\infty}-\langle\tilde{\mathbf{E}}_n\cdot\tilde{\mathbf{H}}\tilde{\mathbf{F}}\rangle=0 \tag{2.23}$$

for all $\tilde{\mathbf{F}}\in\mathscr{V}_\infty$ which is dense in $\mathscr{V}$. ∎

The square integrability of $\tilde{\mathbf{H}}$ is just enough to make sense all the expressions in (2.23). By an abuse of notation, we still denote its closure by $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}$, which is the Friedrichs' extension of a skew-symmetric operator and so is skew-adjoint on $\mathscr{V}_g$.

Because of Lemma 2.1, the space $\mathscr{V}_g(\tilde{\mathbf{H}})$

$$\mathscr{V}_g(\tilde{\mathbf{H}}):=\{\tilde{\mathbf{F}}\in\mathscr{V}_g\,|\,\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{F}}\in\mathscr{V}\} \tag{2.24}$$

is a Hilbert space with the inner product

$$(\tilde{\mathbf{F}},\tilde{\mathbf{G}})_{\tilde{\mathbf{H}}}:=\langle\tilde{\mathbf{F}}\cdot\tilde{\mathbf{G}}\rangle+\langle\tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\mathbf{G}}\rangle,\qquad\forall\tilde{\mathbf{F}},\tilde{\mathbf{G}}\in\mathscr{V}_g(\tilde{\mathbf{H}}). \tag{2.25}$$

The norm associated with the inner product $(\ ,\ )_{\tilde{\mathbf{H}}}$ is denoted by $\|\cdot\|_{\tilde{\mathbf{H}}}$, i.e.,

$$\|\mathbf{F}\|^2_{\tilde{\mathbf{H}}}:=\langle\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}\rangle+\langle\tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\mathbf{F}}\cdot\tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\mathbf{F}}\rangle,\qquad\forall\tilde{\mathbf{F}}\in\mathscr{V}_g(\tilde{\mathbf{H}}). \tag{2.26}$$

Clearly $\mathscr{V}_g(\tilde{\mathbf{H}})$ is a proper subspace of $\mathscr{V}_g$ unless $\tilde{\mathbf{H}}$ is uniformly bounded.

3. THE BOUNDARY VALUE PROBLEM

For simplicity we set σ to be one.

Dropping the hat and λ from $\hat{T}_\varepsilon(\lambda,\mathbf{x},\omega)$ in (1.10) the Dirichlet problem has the weak form

$$\int_D (\mathbf{I}+\mathbf{H}(\mathbf{x}/\varepsilon,\omega))\,\nabla T_\varepsilon(\mathbf{x},\omega)\cdot\nabla\phi(\mathbf{x})\,d\mathbf{x}+\lambda\int_D T_\varepsilon(\mathbf{x},\omega)\,\phi(\mathbf{x})$$

$$=\int_D T_0(\mathbf{x})\,\phi(\mathbf{x})\,d\mathbf{x} \tag{3.1}$$

for every test function $\phi\in\mathscr{C}_0^1(D)$.

We construct solutions of (3.1) by a truncation argument: First we introduce the level M truncation $\tilde{\mathbf{H}}^{(M)}$ of the stream matrix:

$$\tilde{\mathbf{H}}_{i,j}^{(M)}=\begin{cases}\tilde{\mathbf{H}}_{i.j}, & \text{for } |\tilde{\mathbf{H}}_{i,j}|<M\\ \operatorname{sig}(\tilde{\mathbf{H}}_{i,j})M, & \text{for } |\tilde{\mathbf{H}}_{i,j}|\geqslant M\end{cases} \tag{3.2}$$

for all i, j. Thus $|\tilde{\mathbf{H}}_{i,j}^{(M)}|\leqslant M$, $\forall i, j$.

By the individual ergodic theorem [5], the space averages converge to the ensemble average

$$\frac{\varepsilon^d}{|D|}\int_{D/\varepsilon}\mathbf{H}_{i,j}^2(\mathbf{x},\omega)=\frac{1}{|D|}\int_D \mathbf{H}_{i,j}^2\left(\frac{\mathbf{x}}{\varepsilon},\omega\right)\to\langle\tilde{\mathbf{H}}_{i,j}^2\rangle \tag{3.3}$$

as $\varepsilon\to 0$ for almost all realizations. Without loss of generality we may assume that $\mathbf{H}(\mathbf{x},\omega)$ is locally square integrable for almost all realizations. Thus the truncated stream matrix converges to $\mathbf{H}(\mathbf{x},\omega)$ locally in the L^2 sense.

We consider the similar boundary value problem (3.1) associated with the truncated stream matrix $\tilde{\mathbf{H}}^{(M)}$, namely,

$$\int_D (\mathbf{I}+\mathbf{H}^{(M)}(\mathbf{x}/\varepsilon,\omega))\,\nabla T_\varepsilon^{(M)}(\mathbf{x},\omega)\cdot\nabla\phi(\mathbf{x})\,d\mathbf{x}+\lambda\int_D T_\varepsilon^{(M)}(\mathbf{x},\omega)\,\phi(\mathbf{x})$$

$$=\int_D T_0(\mathbf{x})\,\phi(\mathbf{x},\omega)\,d\mathbf{x} \tag{3.4}$$

for any test function ϕ in $\mathscr{C}_0^1(D)$. In fact, the space of test functions for (3.4) can be enlarged to include $H_0^1(D)$ functions but we will not be able to pass to the limit with the latter class of test functions.

We note that the left side of (3.4) defines a bounded coercive bilinear form on $H_0^1(D)$ for almost all ω. On the other hand, the right side of (3.4) defines a bounded linear functional on $H_0^1(D)$ for $T_0\in L^2(D)$. Thus by the Lax–Milgram lemma, (3.4) has a unique solution in $H_0^1(D)$ for each $M>0$.

Subsituting $\phi = T_\varepsilon^{(M)}$, using the skew-symmetry of $\mathbf{H}^{(M)}$ and applying the Cauchy–Schwartz inequality we obtain

$$\int_D \nabla T_\varepsilon^{(M)} \cdot \nabla T_\varepsilon^{(M)} + \lambda \int_D (T_\varepsilon^{(M)})^2 \leqslant \sqrt{\int_D (T_\varepsilon^{(M)})^2} \sqrt{\int_D T_0^2}. \tag{3.5}$$

Applying the Poincaré inequality and solving the quadratic inequality we get the uniform bound

$$\int_D \nabla T_\varepsilon^{(M)} \cdot \nabla T_\varepsilon^{(M)} + \lambda \int_D (T_\varepsilon^{(M)})^2 \leqslant C \int_D T_0^2 \tag{3.6}$$

with the Poincaré constant $C > 0$ independent of M, ε and T_0. Thus there is a weakly convergent subsequence, denoted by $T_\varepsilon^{(M')}$, with which we will pass to the limit $M \to \infty$ in (3.4). Let the weak limit of $T_\varepsilon^{(M')}$ be denoted by T_ε. Note that $\mathbf{H}^{(M)}$ tends to $\mathbf{H}$ strongly in $L^2(D)$ for almost all ω. Thanks to the restriction of test functions to $\mathscr{C}_0^1(D; \mathscr{H})$ we can pass to the limit $M \to \infty$ in (3.4) and obtain a solution for (3.1) for almost all ω. Furthermore, passing the limit in inequality (3.6) we get the *energy inequality*

$$\int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon + \lambda \int_D T_\varepsilon^2 \leqslant C \int_D T_0^2. \tag{3.7}$$

Namely, the energy bound is scale independent. Any families of solutions T_ε of (3.1) satisfying the energy inequality (3.7) with the constant C independent of ε are said to have *uniformly bounded energy*.

When the stream matrix $\tilde{\mathbf{H}}$ is bounded, it is known that the solution of (3.1) satisfying the energy estimate (3.7) is classical and unique (cf. Ladyzhenskaya and Uralćeva [13]). In the case of square integrable stream matrices, the uniqueness for weak solution of (3.1) is a nontrivial issue. In contrast, Fannjiang and Papanicolaou [8] construct a solution in a stronger sense and the solution is unique for all $\varepsilon > 0$ almost surely.

In the present paper we deal with the limits of any weak solutions of (3.1) with the energy estimate (3.7).

A weak solution T_ε of (3.1) is said to satisfy the *energy equality* if

$$\int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon + \lambda \int_D T_\varepsilon^2 = \int_D T_0 T_\varepsilon. \tag{3.8}$$

Clearly (3.8) implies (3.7). The solution produced by the variational methods in ([8]) always satisfies the energy equality (3.8). We do not know if the solution produced by the truncation argument satisfies (3.8). For the solutions satisfying the energy equality (3.8) the corrector result (1.2) holds.

4. THE ABSTRACT CELL PROBLEM

On the probability space $(\Omega, \mathscr{F}, P)$ let us consider the abstract cell problem: For each $k=1, ..., d$ find a gradient field $\tilde{\mathbf{E}}_k \in \mathscr{V}_g(\tilde{\mathbf{H}})$ which has zero mean

$$\langle \tilde{\mathbf{E}}_k \rangle = 0 \tag{4.1}$$

and satisfies the equation

$$\langle \tilde{\mathbf{E}}_k \cdot \tilde{\mathbf{F}} \rangle + \langle \tilde{\boldsymbol{\Gamma}} \tilde{\mathbf{H}} (\tilde{\mathbf{E}}_k + \boldsymbol{e}_k) \cdot \tilde{\mathbf{F}} \rangle = 0, \qquad \forall \tilde{\mathbf{F}} \in \mathscr{V}_g \tag{4.2}$$

where the space $\mathscr{V}_g(\tilde{\mathbf{H}})$ is defined in (2.24).

We define the *effective diffusivity* $\sigma^*_{i,j}$ as

$$\sigma^*_{i,j} := \langle (\mathbf{I} + \tilde{\mathbf{H}})(\boldsymbol{e}_i + \tilde{\mathbf{E}}_i) \cdot \boldsymbol{e}_j \rangle, \qquad i, j = 1, 2, ..., d \tag{4.3}$$

$$= \delta_{i,j} + \langle \tilde{\mathbf{H}} \tilde{\mathbf{E}}_i \cdot \boldsymbol{e}_j \rangle \tag{4.4}$$

The connection between the cell problem and homogenization follows from the usual multiple scale arguments. The cell problem is formally the same in the random as in the periodic case [3, 10]. On physical grounds, the cell problem can be understood as macroscopic concentration gradients $\boldsymbol{e}_k$ that induce through the flow microscopic concentration fluctuations χ_k which in turn lead to enhanced fluxes $-(\mathbf{I} + \mathbf{H}) \nabla \chi_k$ by Fourier's law. The average of the enhanced flux is the macroscopic diffusivity (4.3).

We shall show in the Appendix A the existence and uniqueness of the cell problem:

THEOREM 4.1. *There exist unique solution* $\tilde{\mathbf{E}}_k \in \mathscr{V}_g(\tilde{\mathbf{H}})$, $k = 1, 2, ..., d$ *to the abstract cell problem* (4.1)–(4.2)

Similarly, the adjoint cell problem

$$\langle \tilde{\mathbf{E}}'_k \cdot \tilde{\mathbf{F}} \rangle - \langle \tilde{\boldsymbol{\Gamma}} \tilde{\mathbf{H}} (\tilde{\mathbf{E}}'_k + \boldsymbol{e}_k) \cdot \tilde{\mathbf{F}} \rangle = 0, \qquad \forall \tilde{\mathbf{F}} \in \mathscr{V}_g, \quad \forall \tilde{\mathbf{F}} \in \mathscr{V}_g \tag{4.5}$$

$$\langle \mathbf{E}'_k \rangle = 0 \tag{4.6}$$

admits a unique solution $\tilde{\mathbf{E}}'_k \in \mathscr{V}_g(\tilde{\mathbf{H}})$.

The effective diffusivity σ^* *defined by* (4.3) *equals*

$$\sigma^*_{i,j} = \langle (\mathbf{I} - \tilde{\mathbf{H}})(\boldsymbol{e}_j + \tilde{\mathbf{E}}'_j) \cdot \boldsymbol{e}_i \rangle, \qquad i, j = 1, 2, ..., d. \tag{4.7}$$

The primitive function χ_k of the solution $\tilde{\mathbf{E}}_k$ to the *abstract cell problem* (4.1)–(4.2)

$$\nabla \chi_k(\mathbf{x}, \omega) = \mathbf{E}_k(\mathbf{x}, \omega), \qquad i = 1, 2, 3, ..., d \tag{4.8}$$

is called the *corrector*. The primitive function χ'_k of the solution $\tilde{\mathbf{E}}'_k$ to the *adjoint cell problem* (4.5)–(4.6) is called the *adjoint corrector*. They are nonstationary and are unique when normalized by

$$\chi_k(0, \omega) = 0, \qquad i = 1, 2, 3, ..., d \tag{4.9}$$

$$\chi'_k(0, \omega) = 0, \qquad i = 1, 2, 3, ..., d \tag{4.10}$$

for almost all ω.

It can be shown (cf. Lemma A.1 that the unique solution to the abstract cell problem is the weak limit in $\mathscr{V}_g$ of the sequence of solutions to the truncated cell problem:

$$\langle (\mathbf{I} + \tilde{\mathbf{H}}^{(M)})(\boldsymbol{e}_k + \tilde{\mathbf{E}}_k^{(M)}) \cdot \tilde{\mathbf{F}} \rangle = 0, \qquad \forall \tilde{\mathbf{F}} \in \mathscr{V}_g \tag{4.11}$$

$$\langle \tilde{\mathbf{E}}_k^{(M)} \rangle = 0 \tag{4.12}$$

where the stream matrix $\tilde{\mathbf{H}}^{(M)}$ is the level M truncation, defined by (3.2), of the original stream matrix $\tilde{\mathbf{H}}$.

Now we state the crucial estimate used in the proof of the main theorem. It says, in essence, that the cell problem and its adjoint can be solved approximately in the appropriate sense in the space $\mathscr{B}_g$ of bounded gradient fields.

LEMMA 4.1. *Given $\delta > 0$ there exists bounded gradient field $\tilde{\mathbf{G}}_i$ and $\tilde{\mathbf{F}}_i$ in $\mathscr{B}_g$ such that*

$$\langle \tilde{\mathbf{G}} \rangle = 0 \tag{4.13}$$

$$\langle \tilde{\mathbf{F}} \rangle = 0 \tag{4.14}$$

$$\langle |\tilde{\mathbf{G}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i)|^2 \rangle < \delta^2 \tag{4.15}$$

$$\langle |\tilde{\mathbf{F}}_i - \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{F}}_i + \boldsymbol{e}_i)|^2 \rangle < \delta^2 \tag{4.16}$$

$$|\langle (\mathbf{I} + \tilde{\mathbf{H}})(\boldsymbol{e}_i + \tilde{\mathbf{G}}_i)\, \boldsymbol{e}_j \rangle - \sigma^*_{i,j}| < \delta \sqrt{\langle \mathbf{E}'_j \cdot \tilde{\mathbf{E}}'_j \rangle} \tag{4.17}$$

$$|\langle (\mathbf{I} - \tilde{\mathbf{H}})(\boldsymbol{e}_i + \tilde{\mathbf{F}}_i)\, \boldsymbol{e}_j \rangle - \sigma^*_{j,i}| < \delta \sqrt{\langle \tilde{\mathbf{E}}_j \cdot \tilde{\mathbf{E}}_j \rangle}. \tag{4.18}$$

Here $\tilde{\mathbf{E}}'_j$ and $\tilde{\mathbf{E}}_j$ are as given in Theorem 4.1.

The proof of Lemma 4.1 is given in the Appendix.

Naturally we call the primitive functions $g_i(\mathbf{x}, \omega)$ and $f_i(\mathbf{x}, \omega)$ of bounded gradient fields $\tilde{\mathbf{G}}_i$ and $\tilde{\mathbf{F}}_i$ in Lemma 5 *approximate correctors* and *approximate adjoint correctors*, respectively.

For the approximate correctors and the adjoints we have the following L^∞ bound.

Lemma 4.2. *Let $\tilde{\mathbf{F}} \in \mathscr{C}_g$ has zero mean and let $\nabla f(\mathbf{x}, \omega) = \nabla f(\mathbf{x}, \omega)$, $f(0, \omega) = 0$. Then, $f(\mathbf{x}, \omega)$ agrees with a continuously differentiable function in $\mathbf{x}$ for almost all ω. Moreover,*

$$\lim_{\varepsilon \to 0} \varepsilon f(\mathbf{x}/\varepsilon, \omega) = 0 \tag{4.19}$$

in the space $\mathscr{C}(D)$ of continuous functions.

Proof. The first part of the lemma is evident. Let us turn to the convergence statement.

Set $f^\varepsilon(\mathbf{x}, \omega) = \varepsilon f(\mathbf{x}/\varepsilon, \omega)$. By definition,

$$\nabla f^\varepsilon(\mathbf{x}, \omega) = \tilde{\mathbf{F}}(\tau_{-\mathbf{x}/\varepsilon} \omega) \in \mathscr{C}_g. \tag{4.20}$$

Hence f^ε 's uniformly bounded on D:

$$|f^\varepsilon(\mathbf{x}, \omega)| \leqslant \left| \int_0^{\mathbf{x}} \nabla f^\varepsilon(\mathbf{y}, \omega) \cdot d\mathbf{y} \right| \leqslant C\, |\mathbf{x}|. \tag{4.21}$$

Thus it follows from (4.20) and (4.21) that f^ε is uniformly bounded in the space $\mathscr{C}^1(D)$ for almost all ω. By the compact imbedding

$$\mathscr{C}^1(D) \hookrightarrow \mathscr{C}(D) \tag{4.22}$$

there is a convergent subsequence, still denoted by f^ε, in $\mathscr{C}(D)$.

On the other hand, $\{\nabla f^\varepsilon\}$ is precompact in the weak-star topology in $(L^1{}^*(D))^d$. The limit can be identified by applying the averaging Lemma 4.3:

$$\lim_{\varepsilon \to 0} \int_D \tilde{\mathbf{F}}(\tau_{-\mathbf{x}/\varepsilon} \omega)\, \mathbf{G}(\mathbf{x}) = \langle \tilde{\mathbf{F}} \rangle \int_D \mathbf{G} = 0 \tag{4.23}$$

for almost all ω.

Therefore the convergent subsequence f^ε must also converge to a constant in view of (4.23). But this constant must be zero by the condition $f^\varepsilon(0, \omega) = 0$. ∎

We now state the averaging lemma used in the proofs of Lemma 4.2 and Theorem 1.1.

Lemma 4.3. *Suppose $\tilde{f}(\omega) \in \mathscr{H}$ and $\phi(\mathbf{x}) \in L^2(D)$. Then*

$$\int_D U_{\mathbf{x}/\varepsilon} \tilde{f}(\omega)\, \phi(\mathbf{x}) \to \langle \tilde{f} \rangle \int_D \phi \tag{4.24}$$

for almost all ω.

Proof. The lemma holds in case when $\phi(\mathbf{x})$ is a characteristic function of an interval since that is in essence the statement of the Individual Ergodic Theorem for a multiparameter commuting group of contractions (cf. Dunford and Schwartz [6]). We can therefore easily generalize the conclusion of the lemma to the linearly dense subset L of $L^2(D)$ consisting of all finite linear combinations of such functions. Now let $\phi \in L^2(D)$ and $\delta > 0$ be chosen arbitrarily. Let $\tilde{\phi} \in L$ and

$$\|\phi - \tilde{\phi}\|_{L^2(D)} < \delta. \tag{4.25}$$

We can write then that

$$\begin{aligned} &\left| \int_D U_{\mathbf{x}/\varepsilon} \tilde{f}(\omega)\, \phi(\mathbf{x}) - \langle \tilde{f} \rangle \int_D \phi \right| \\ &\quad \leqslant \left[\int_D |U_{\mathbf{x}/\varepsilon} \tilde{f}|^2 (\omega) \right]^{1/2} \|\phi - \tilde{\phi}\|_{L^2(D)} \\ &\quad\quad + \left| \int_D U_{\mathbf{x}/\varepsilon} \tilde{f}(\omega)\, \tilde{\phi}(\mathbf{x}) - \langle \tilde{f} \rangle \int_D \tilde{\phi} \right| + \langle |\tilde{f}| \rangle \, \|\phi - \tilde{\phi}\|_{L^1(D)}. \end{aligned}$$

Allowing $\varepsilon \downarrow 0$ we obtain, thanks to the Individual Ergodic Theorem and (4.25), that

$$\limsup_{\varepsilon \downarrow 0} \left| \int_D U_{\mathbf{x}/\varepsilon} \tilde{f}(\omega)\, \phi(\mathbf{x}) - \langle \tilde{f} \rangle \int_D \phi \right| \leqslant \delta\, |D|^{1/2} \left(\|\tilde{f}\|_{L^2(D)} + \|\tilde{f}\|_{L^1(\Omega)} \right).$$

Since $\delta > 0$ was chosen arbitrarily this implies the lemma. ∎

5. PROOF OF THEOREM 1.1: HOMOGENIZATION

We modify Tartar's argument ([15, 14]) with the use of approximate correctors.

By the condition (3.7) of uniformly bounded energy the solution T_ε is pre-compact weakly in $H_0^1(D, \mathscr{H})$ and pre-compact strongly in $L^2(D, \mathscr{H})$. Extracting a convergent subsequence in both spaces, still denoted by T_ε, and let $\bar{T}$ be the limit.

Define the flux

$$\mathbf{Q}_\varepsilon := -(\mathbf{I} + \mathbf{H}_\varepsilon)\, \nabla T_\varepsilon \tag{5.1}$$

with

$$\mathbf{H}_\varepsilon(\mathbf{x}, \omega) = \mathbf{H}(\mathbf{x}/\varepsilon, \omega). \tag{5.2}$$

The flux $\mathbf{Q}_\varepsilon$ is uniformly bounded in $(L^1(D))^d$ for almost all ω since both $\mathbf{H}_\varepsilon$ and ∇T_ε are uniformly bounded in $(L^2(D))^d$ for almost all ω. We shall regard $L^1(D)$ as a subspace of $M(D)$, the space of Radon measures on D. By Helly's selection theorem, $\mathbf{Q}_\varepsilon$ is pre-compact in the weak-star topology of $(M(D))^d$. Extracting a convergent subsequence, still denoted by $\mathbf{Q}_\varepsilon$, and passing to the limit we obtain some limit flux $\bar{\mathbf{Q}}(d\mathbf{x}) \in (M(D))^d$, which is a finite, vector-valued Radon measure.

The remaining question for homogenization is to identify the relation between the limit solution $\nabla \bar{T}$ and the limit flux $\bar{\mathbf{Q}}$. The goal is to show that $\bar{\mathbf{Q}}$ is absolutely continuous with respect to the Lebesque measure and is linearly related to $\nabla \bar{T}$ with the proportionality given by σ^*.

Passing to the limit in Eq. (3.1) we obtain

$$-\int_D \bar{\mathbf{Q}} \cdot \nabla\phi + \lambda \int_D \bar{T}\phi = \int_D T_0 \phi \tag{5.3}$$

for any $\phi \in \mathscr{C}_0^1(D)$. This is the step which would not go through for the bigger test function space such as $W_0^{1,\infty}(D)$.

Let $f_i(\mathbf{x}, \omega)$ be an approximate adjoint corrector

$$\nabla f_i(\mathbf{x}, \omega) = \mathbf{F}_i(\mathbf{x}, \omega) \tag{5.4}$$

$$f_i(0, \omega) = 0 \tag{5.5}$$

with $\mathbf{F}_i(\mathbf{x}, \omega) = \tilde{\mathbf{F}}(\tau_{-\mathbf{x}}\omega) \in (\mathscr{C}(D))^d$ as stated in Lemma 4.1. Consider the test function

$$w_i(\mathbf{x}, \omega) = \mathbf{x} \cdot \boldsymbol{e}_i + f_i(\mathbf{x}, \omega) \tag{5.6}$$

and the scaled version

$$w_i^\varepsilon(\mathbf{x}, \omega) = \varepsilon w_i(\mathbf{x}/\varepsilon, \omega) = \mathbf{x} \cdot \boldsymbol{e}_i + \varepsilon f_i(\mathbf{x}/\varepsilon, \omega). \tag{5.7}$$

By Lemma 4.2, both w_i and w_i^ε are in the space $\mathscr{C}(D)$.

From inequality (4.16) in Lemma 4.1 and the individual ergodic theorem it follows that

$$\int_D [\nabla w_i^\varepsilon - \boldsymbol{e}_i - \boldsymbol{\Gamma}\mathbf{H}_\varepsilon \nabla w_i^\varepsilon]^2 \leqslant \delta^2 \tag{5.8}$$

for sufficiently small ε, almost all ω and any given $\delta > 0$. Applying the Cauchy–Schwartz inequality we have that

$$\left|\int_D (\mathbf{I}-\mathbf{H}_\varepsilon)\,\nabla w_i^\varepsilon\cdot\nabla\phi\right| \tag{5.9}$$

$$=\left|\int_D [(\mathbf{I}-\Gamma\mathbf{H}_\varepsilon)\,\nabla w_i^\varepsilon-\boldsymbol{e}_i]\cdot\nabla\phi\right.$$

$$<\delta\sqrt{\int_D |\nabla\phi|^2} \tag{5.10}$$

for any $\phi\in\mathscr{C}_0^1(D)$. Note here that $\int_D \boldsymbol{e}_i\cdot\nabla\phi=0$ for ϕ with vanshing boundary data.

Let $\theta(\mathbf{x},\omega)$ be any function in $\mathscr{C}_0^\infty(D)$. It is clear that both θT_ε and θw_i^ε are admissible test functions for (5.9) and (3.1) respectively. Inserting them respectively we have

$$\left|\int_D (\mathbf{I}+\mathbf{H}_\varepsilon)\,\nabla(\theta T_\varepsilon)\cdot\nabla w_i^\varepsilon\right|<\delta\sqrt{\int_D |\nabla(\theta T_\varepsilon)|^2} \tag{5.11}$$

$$-\int_D \boldsymbol{Q}_\varepsilon\cdot\nabla(\theta w_i^\varepsilon)+\lambda\int_D T_\varepsilon\theta w_i^\varepsilon=\int_D T_0\theta w_i^\varepsilon. \tag{5.12}$$

Subtracting (5.11) from (5.12) we obtain

$$\left|\int_D T_\varepsilon\nabla\theta\cdot(\mathbf{I}-\mathbf{H}_\varepsilon)\,\nabla w_i^\varepsilon-\int_D w_i^\varepsilon\boldsymbol{Q}_\varepsilon\cdot\nabla\theta+\lambda\int_D T_\varepsilon\theta w_i^\varepsilon-\int_D T_0\theta w_i^\varepsilon\right|$$

$$\leqslant\delta\sqrt{\int_D |\nabla(\theta T_\varepsilon)|^2}\leqslant C_1\delta \tag{5.13}$$

where the constant C_1 depends on the energy bound C in (3.7) and the function θ. Lemma 4.19 (regarding the asymptotic behavior of w_i^ε) and the strong convergence of T_ε to $\bar{T}$ in $L^2(D)$ allow us to pass to the limit in inequality (5.13):

$$\int_D T_\varepsilon\,\nabla\theta(\mathbf{I}-\mathbf{H}_\varepsilon)\,\nabla w_i^\varepsilon\to\int_D \bar{T}\,\nabla\theta\cdot\langle(\mathbf{I}-\tilde{\mathbf{H}})(\boldsymbol{e}_i+\tilde{\mathbf{F}}_i)\rangle \tag{5.14}$$

$$\int_D w_i^\varepsilon\boldsymbol{Q}_\varepsilon\cdot\nabla\theta\to\int_D \bar{\boldsymbol{Q}}\cdot\nabla\theta\mathbf{x}\cdot\boldsymbol{e}_i \tag{5.15}$$

$$\lambda\int_D T_\varepsilon\theta w_i^\varepsilon\to\lambda\int_D \bar{T}\theta\mathbf{x}\cdot\boldsymbol{e}_i \tag{5.16}$$

$$\int_D T_0\theta w_i^\varepsilon\to\int_D T_0\theta\mathbf{x}\cdot\boldsymbol{e}_i \tag{5.17}$$

and it follows that

$$\left|-\int_D \bar{T}\langle(\mathbf{I}-\tilde{\mathbf{H}})(\boldsymbol{e}_i+\tilde{\mathbf{F}}_i)\rangle\cdot\nabla\theta-\int_D \bar{\mathbf{Q}}\cdot\nabla\theta\mathbf{x}\cdot\boldsymbol{e}_i+\lambda\int_D \bar{T}\theta\mathbf{x}\cdot\boldsymbol{e}_i-\int_D T_0\theta\mathbf{x}\cdot\boldsymbol{e}_i\right|$$
$$\leqslant C_1\delta \tag{5.18}$$

for any $\delta>0$.

The convergence (5.14) is justified by Lemma 4.3 and the strong convergence of T_ε in $L^2(D)$. The convergence of (5.15)–(5.17) is justified by the convergence of w_i^ε to $\mathbf{x}\cdot\boldsymbol{e}_i$ in the space $\mathscr{C}_0(D)$ (Lemma A.1).

In view of (4.18) in Lemma 4.1, we let δ tend to zero and obtain the equality from (5.18)

$$-\int_D \bar{T}\sum_j \frac{\partial\theta}{\partial x_j}\sigma^*_{j,i}-\int_D \bar{\boldsymbol{Q}}\cdot\nabla\theta\mathbf{x}\cdot\boldsymbol{e}_i+\lambda\int_D \bar{T}\theta\mathbf{x}\cdot\boldsymbol{e}_i=\int_D T_0\theta\mathbf{x}\cdot\boldsymbol{e}_i. \tag{5.19}$$

Inserting $\phi=\theta\mathbf{x}\cdot\boldsymbol{e}_i$ in the Eq. (5.3) we also have that

$$-\int_D \bar{\mathbf{Q}}\cdot\boldsymbol{e}_i\theta-\int_D \bar{\mathbf{Q}}\cdot\nabla\theta\mathbf{x}\cdot\boldsymbol{e}_i+\lambda\int_D \bar{T}\theta\mathbf{x}\cdot\boldsymbol{e}_i=\int_D T_0\theta\mathbf{x}\cdot\boldsymbol{e}_i. \tag{5.20}$$

Subtracting (5.19) from (5.20) yields

$$\int_D \sum_j \sigma^*_{j,i}\frac{\partial\theta}{\partial x_j}\bar{T}=\int_D \bar{\mathbf{Q}}\cdot\boldsymbol{e}_i\theta \tag{5.21}$$

for any $\theta\in\mathscr{C}_0^\infty(D)$. Because $\mathscr{C}_0^\infty(D)$ is dense in $\mathscr{C}_0(D)$, (5.21) identifies the limit flus $\bar{\mathbf{Q}}$, after integration by parts, as a L^2-function

$$-\sum_j \sigma^*_{j,i}\frac{\partial\bar{T}}{\partial x_j}=\bar{\mathbf{Q}}\cdot\boldsymbol{e}_i. \tag{5.22}$$

Inserting the identity (5.22) into Eq. (5.3) gives

$$\int_D \sum_{i,j}\sigma^*_{i,j}\frac{\partial\bar{T}}{\partial x_i}\frac{\partial\phi}{\partial x_j}+\lambda\int_D \bar{T}\phi=\int_D T_0\phi \tag{5.23}$$

for all $\phi\in\mathscr{C}^1(D)$. Restricting to $\phi\in\mathscr{C}_c^\infty(D)$ and integrating by parts, we see that only the symmetric part of $\sigma^*_{i,j}$ contributes to the first integral. So we rewrite Eq. (5.23) in the symmetric form

$$\int_D \sum_{i,j}\frac{1}{2}(\sigma^*_{i,j}+\sigma^*_{j,i})\frac{\partial\bar{T}}{\partial x_i}\frac{\partial\phi}{\partial x_j}+\lambda\int_D \bar{T}\phi=\int_D T_0\phi \tag{5.24}$$

for $\phi\in\mathscr{C}_c^\infty(D)$.

The proof of Theorem 1.1 is now complete. ▮

6. PROOF OF THEOREM 1.2: TWO-TERM ASYMPTOTICS

Notice that for $T_0 \in \mathscr{C}^\infty(D)$ the solution $\bar{T}$ of (5.24) is also in the space $\mathscr{C}^\infty(D)$. All we need here is the boundedness of the second derivatives of $\bar{T}$.

For any $\delta > 0$ let $g_i(\mathbf{x}, \omega)$ be the approximate correctors with gradients $\nabla g_i(\mathbf{x}, \omega) = \mathbf{G}_i(\mathbf{x}, \omega)$ as asserted in Lemma 4.1. Then by Lemma 4.3 and 4.1 we have that

$$\lim_{\varepsilon \to 0} \int_D \left(\sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) (\nabla \chi_i(\mathbf{x}/\varepsilon) - \nabla g_i(\mathbf{x}/\varepsilon)) \right)^2$$

$$= \lim_{\varepsilon \to 0} \int_D \left(\sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) (\mathbf{E}_i(\mathbf{x}/\varepsilon) - \mathbf{G}_i(\mathbf{x}/\varepsilon)) \right)^2$$

$$= \int_D \left(\sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) \langle (\tilde{\mathbf{E}}_i - \tilde{\mathbf{G}}_i) \rangle \right)^2 \leqslant d\delta^2 \int_D \sum_i \left(\frac{\partial \bar{T}}{\partial x_i} \right)^2. \tag{6.1}$$

Thus, to prove Theorem 1.2 it is sufficient to show that

$$\limsup_{\varepsilon \to 0} \int_D \left(\nabla T_\varepsilon(\mathbf{x}) - \nabla \bar{T}(\mathbf{x}) - \sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) \, \nabla g_i(\mathbf{x}/\varepsilon) \right)^2 \leqslant c\delta \tag{6.2}$$

for some $c > 0$ independent of δ.

Due to the skew symmetry of $\mathbf{H}_\varepsilon$ the integral on the left side of (6.2) is equal to

$$\int_D \left(\nabla T_\varepsilon(\mathbf{x}) - \nabla \bar{T}(\mathbf{x}) - \sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) \, \nabla g_i(\mathbf{x}/\varepsilon) \right)^2$$

$$= \int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon - \int_D (\mathbf{I} + \mathbf{H}_\varepsilon(\mathbf{x})) \, \nabla T_\varepsilon(\mathbf{x}) \cdot \left(\nabla \bar{T}(\mathbf{x}) + \sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) \, \nabla g_i(\mathbf{x}/\varepsilon) \right)$$

$$- \int_D \nabla T_\varepsilon(\mathbf{x}) \cdot (\mathbf{I} + \mathbf{H}_\varepsilon(\mathbf{x})) \left(\nabla \bar{T}(\mathbf{x}) + \sum_i \frac{\partial \bar{T}}{\partial x_i} (\mathbf{x}) \, \nabla g_i(\mathbf{x}/\varepsilon) \right)$$

$$+ \int_D \sum_{i,j} \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \bar{T}}{\partial x_j} (\mathbf{G}_i(\mathbf{x}/\varepsilon) + \boldsymbol{e}_i) \cdot (\mathbf{G}_j(\mathbf{x}/\varepsilon) + \boldsymbol{e}_j). \tag{6.3}$$

Note here that $\nabla \bar{T}$ is bounded on D so the integrals are well defined.

By the assumed energy equality and the previously proved strong convergence of T_ε the first integral in (6.3) becomes, in the limit $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon = \lim_{\varepsilon \to 0} \left[\int_D T_0 T_\varepsilon - \lambda \int_D T_\varepsilon^2 \right] = \int_D T_0 \bar{T} - \lambda \int_D \bar{T}^2. \tag{6.4}$$

Rewrite the second integral in (6.3) as

$$\int_D (\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))\,\nabla T_\varepsilon(\mathbf{x})\cdot\left(\nabla\bar{T}(\mathbf{x})+\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})\,\nabla g_i(\mathbf{x}/\varepsilon)\right)$$
$$=\int_D (\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))\,\nabla T_\varepsilon(\mathbf{x})\cdot\nabla\left(\bar{T}(\mathbf{x})+\varepsilon\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})\,g_i(\mathbf{x}/\varepsilon)\right)$$
$$-\int_D (\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))\,\nabla T_\varepsilon(\mathbf{x})\cdot\varepsilon\sum_i \frac{\partial \nabla\bar{T}}{\partial x_i}(\mathbf{x})\,g_i(\mathbf{x}/\varepsilon). \tag{6.5}$$

By (3.1) and (5.1), (6.5) now becomes

$$-\lambda\int_D T_\varepsilon(\mathbf{x})\left(\bar{T}(\mathbf{x})+\varepsilon\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})\,g_i(\mathbf{x}/\varepsilon)\right)$$
$$+\int_D T_0(\mathbf{x})\left(\bar{T}(\mathbf{x})+\varepsilon\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})\,g_i(\mathbf{x}/\varepsilon)\right)$$
$$+\int_D \boldsymbol{Q}_\varepsilon(\mathbf{x})\cdot\varepsilon\sum_i \frac{\partial \nabla\bar{T}}{\partial x_i}(\mathbf{x})\,g_i(\mathbf{x}/\varepsilon). \tag{6.6}$$

Note that $\partial\bar{T}/(\partial x_i\,\partial x_j)$ is bounded on D so the last integral is well defined.

Passing to the limit in (6.6) with the strong convergence of T_ε and Lemma 4.19 we have that

$$\lim_{\varepsilon\to 0}\int_D (\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))\,\nabla T_\varepsilon(\mathbf{x})\cdot\left(\nabla\bar{T}(\mathbf{x})+\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})\,\nabla g_i(\mathbf{x}/\varepsilon)\right)$$
$$=-\lambda\int_D \bar{T}^2+\int_D T_0\bar{T}. \tag{6.7}$$

The third integral in (6.3) can be written as

$$\int_D \nabla T_\varepsilon(\mathbf{x})\cdot\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})(\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon))$$
$$=\sum_i\int_D \nabla\left(T_\varepsilon\frac{\partial \bar{T}}{\partial x_i}\right)(\mathbf{x})\cdot(\mathbf{I}+\mathbf{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon))$$
$$-\sum_i\int_D T_\varepsilon\frac{\partial\,\nabla\bar{T}}{\partial x_i}(\mathbf{x})\cdot(\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon)) \tag{6.8}$$

By Lemma 4.1, we have the inequality, similar to (5.9),

$$\left|\int_D (\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon))\cdot\nabla\phi\right|\leqslant\delta\sqrt{\int_D |\nabla\phi|^2} \tag{6.9}$$

for all $\phi \in \mathscr{C}_0^1(D)$ and sufficiently small ε. Consequently,

$$\left|\sum_i \int_D \nabla\left(T_\varepsilon \frac{\partial \bar{T}}{\partial x_i}\right)(\mathbf{x})\cdot(\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon))\right|$$
$$\leqslant \delta \sum_i \sqrt{\int_D \left|\nabla\left(T_\varepsilon \frac{\partial \bar{T}}{\partial x_i}\right)\right|^2} \leqslant c_1 \delta \tag{6.10}$$

due to the energy bound (3.7) and the boundedness of the second derivatives of $\bar{T}$.

Passing the limit in the second term in (6.8) with the strong convergence of T_ε and Lemma 4.3 we have that

$$\lim_{\varepsilon\to 0} \sum_i \int_D T_\varepsilon \frac{\partial \nabla \bar{T}}{\partial x_i}(\mathbf{x})\cdot(\mathbf{I}+ -\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon))$$
$$= \sum_i \int_D \bar{T} \frac{\partial \nabla \bar{T}}{\partial x_i}(\mathbf{x})\cdot\langle(\mathbf{I}+\tilde{\mathbf{H}})(\boldsymbol{e}_i+\tilde{\mathbf{G}})\rangle. \tag{6.11}$$

Hence,

$$\left|\lim_{\varepsilon\to 0}\sum_i \int_D T_\varepsilon \frac{\partial \nabla \bar{T}}{\partial x_i}(\mathbf{x})\cdot(\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon)) - \sum_{i,j}\int_D \bar{T}\frac{\partial^2 \bar{T}}{\partial x_i\,\partial x_j}\sigma^*_{i,j}\right|$$
$$= \left|\sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}(\mathbf{x})\cdot\langle(\mathbf{I}+\tilde{\mathbf{H}})(\boldsymbol{e}_i+\tilde{\mathbf{G}}_i)\rangle + \int_D \sum_{i,j}\sigma^*_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}\right|$$
$$= \left|\sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}(\mathbf{x})\cdot\langle(\mathbf{I}+\tilde{\mathbf{H}})(\boldsymbol{e}_i+\tilde{\mathbf{G}}_i)\rangle - \lambda\int_D \bar{T}^2 + \int_D T_0\bar{T}\right| \leqslant c_2\delta \tag{6.12}$$

following from Lemma 4.1 and (5.23).

Thus, by (6.8), (6.10) and (6.12) we have

$$\limsup_{\varepsilon\to 0}\left|\int_D \nabla T_\varepsilon(\mathbf{x})\cdot\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})(\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon))\right.$$
$$\left.+\sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}\cdot\langle(\mathbf{I}+\tilde{\mathbf{H}})(\boldsymbol{e}_i+\tilde{\mathbf{G}}_i)\rangle\right|$$
$$\leqslant \limsup_{\varepsilon\to 0}\left|\int_D \nabla T_\varepsilon(\mathbf{x})\cdot\sum_i \frac{\partial \bar{T}}{\partial x_i}(\mathbf{x})\right.$$
$$\left.\times(\mathbf{I}+\mathbf{H}_\varepsilon(\mathbf{x}))(\boldsymbol{e}_i+\mathbf{G}_i(\mathbf{x}/\varepsilon)) + \lambda\int_D \bar{T}^2 - \int_D T_0\bar{T}\right|$$
$$\leqslant c_3\delta. \tag{6.13}$$

For the fourth integral in (6.3) we apply Lemma 4.3 in passing to the limit and Lemma 4.1

$$\lim_{\varepsilon\to 0}\int_D \sum_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}(\mathbf{G}_i(\mathbf{x}/\varepsilon)+\boldsymbol{e}_i)\cdot(\mathbf{G}_j(\mathbf{x}/\varepsilon)+\boldsymbol{e}_j)$$

$$=\int_D \sum_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}\langle(\boldsymbol{e}_i+\tilde{\mathbf{G}}_i)\cdot(\boldsymbol{e}_j+\tilde{\mathbf{G}}_j)\rangle$$

$$=\int_D \sum_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}\langle(\boldsymbol{e}_i+\tilde{\mathbf{E}}_i)\cdot(\boldsymbol{e}_j+\tilde{\mathbf{E}}_j)\rangle$$

$$+\int_D \sum_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}(\langle(\boldsymbol{e}_i+\tilde{\mathbf{G}}_i)\cdot(\boldsymbol{e}_j+\tilde{\mathbf{G}}_j)\rangle-\langle(\mathbf{e}_i+\tilde{\mathbf{E}}_i)\cdot(\boldsymbol{e}_j+\tilde{\mathbf{E}}_j)\rangle)$$

$$=\int_D \sum_{i,j}\frac{1}{2}(\sigma^*_{i,j}+\sigma^*_{j,i})\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}+c_4\delta$$

$$=-\lambda\int_D \bar{T}^2+\int_D T_0\bar{T}+c_4\delta. \tag{6.14}$$

Here we have used also the identity (A.29).

Now (6.2) clearly follows from (6.4), (6.7), (6.13) and (6.14). This concludes the proof. ∎

A. APPENDIX: VARIATIONAL PRINCIPLES

A.1. *The Cell Problem: Proof of Theorem* 4.1

Let $\{\boldsymbol{e}_i\}$ be an orthonormal basis of R^d. We define the continuous, saddle, bi-quadratic functionals $\mathcal{S}_{ij}(\mathbf{F},\mathbf{F}')$, $\forall i, j=1, 2, ..., d$ on the product space $\mathcal{V}_g(\mathbf{H})\otimes\mathcal{V}_g(\mathbf{H})$:

$$\mathcal{S}_{ij}(\tilde{\mathbf{F}},\tilde{\mathbf{F}}'):=\langle\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}\cdot\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}\rangle-2\langle\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}}'\rangle-\langle\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}'\cdot\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}'\rangle+\langle\tilde{\mathbf{F}}\cdot\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i-\boldsymbol{e}_j)\rangle$$
$$-\langle\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i+\boldsymbol{e}_j)\cdot\tilde{\mathbf{F}}'\rangle \tag{A.1}$$

$\forall\tilde{\mathbf{F}},\tilde{\mathbf{F}}'\in\mathcal{V}_g(\tilde{\mathbf{H}})$. It is clear that $\mathcal{S}_{ij}(\tilde{\mathbf{F}},\tilde{\mathbf{F}}')$ is convex in $\tilde{\mathbf{F}}$, concave in $\tilde{\mathbf{F}}'$ and continuous in $\tilde{\mathbf{F}}$, $\tilde{\mathbf{F}}'$.

Let us consider the variational problems:

$$J(\boldsymbol{e}_i,\boldsymbol{e}_j):=\inf_{\tilde{\mathbf{F}}\in\mathcal{V}_g(\tilde{\mathbf{H}})}\ \sup_{\tilde{\mathbf{F}}'\in\mathcal{V}_g(\tilde{\mathbf{H}})}\mathcal{S}_{ij}(\tilde{\mathbf{F}},\tilde{\mathbf{F}}') \tag{A.2}$$

$$K(\boldsymbol{e}_i,\boldsymbol{e}_j):=\sup_{\tilde{\mathbf{F}}'\in\mathcal{V}_g(\tilde{\mathbf{H}})}\ \inf_{\tilde{\mathbf{F}}\in\mathcal{V}_g(\tilde{\mathbf{H}})}\mathcal{S}_{ij}(\tilde{\mathbf{F}},\tilde{\mathbf{F}}'). \tag{A.3}$$

By the von Neumann Minimax Theorem and its generalization (cf. Zeidler [16], Chapter 2.13, Theorem 2.G and Proposition 1), we know

PROPOSITION A.1. *The functional* $\mathscr{S}_{ij}$ *has a saddle point* $(\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j})$ *with respect to* $\mathscr{V}_g(\tilde{\mathbf{H}}) \times \mathscr{V}_g(\tilde{\mathbf{H}})$ *and the relation*

$$\mathscr{S}_{ij}(\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j}) = J(\boldsymbol{e}_i, \boldsymbol{e}_j) = K(\boldsymbol{e}_i, \boldsymbol{e}_j) \tag{A.4}$$

holds true.

By a saddle point $(\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j})$ of $\mathscr{S}_{ij}$ with respect to $\mathscr{V}_g(\tilde{\mathbf{H}}) \times \mathscr{V}_g(\tilde{\mathbf{H}})$ we mean that $\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j} \in \mathscr{V}_g(\tilde{\mathbf{H}})$ and the inequalities hold

$$\mathscr{S}_{ij}(\tilde{\mathbf{E}}_{ij}, \tilde{\mathbf{F}}') \leqslant \mathscr{S}_{ij}(\tilde{\mathbf{E}}_{ij}, \tilde{\mathbf{E}}'_{i,j}) \leqslant \mathscr{S}_{ij}(\tilde{\mathbf{F}}, \tilde{\mathbf{E}}'_{i,j}), \qquad \forall \tilde{\mathbf{F}}, \tilde{\mathbf{F}}' \in \mathscr{V}_g(\tilde{\mathbf{H}}). \tag{A.5}$$

Note that The bilinear form J has the following symmetry property:

$$J(\boldsymbol{e}_i, -\boldsymbol{e}_j) = J(-\boldsymbol{e}_i, \boldsymbol{e}_j) = -J(\boldsymbol{e}_i, \boldsymbol{e}_j). \tag{A.6}$$

For fixed $\tilde{\mathbf{F}} \in \mathscr{V}_g(\tilde{\mathbf{H}})$, the supremum in (A.2) is given by

$$\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}}' = -\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}}\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}} - \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}}\frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \in \mathscr{V}_g. \tag{A.7}$$

$\tilde{\mathbf{F}}'$ is unique if the mean $\langle \tilde{\mathbf{F}}' \rangle$ is specified. In general, $\tilde{\mathbf{F}}'$ is not in the space $\mathscr{V}_g(\tilde{\mathbf{H}})$.

Upon substituting (A.7) in (A.2), we obtain

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) = \inf_{\tilde{\mathbf{F}} \in \mathscr{V}_g(\tilde{\mathbf{H}})} \left\{ \langle \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}} \rangle + \left\langle \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \cdot \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \right\rangle + \langle \tilde{\mathbf{F}} \cdot \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle \right\}. \tag{A.8}$$

Similarly, we eliminate the infimum in (A.3) by solving

$$\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}} = -\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}}\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}}' - \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}}\frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \in \mathscr{V}_g. \tag{A.9}$$

We obtain

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sup_{\tilde{\mathbf{F}}' \in \mathscr{V}_g(\tilde{\mathbf{H}})} \left\{ -\langle \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}}' \cdot \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}}' \rangle - \left\langle \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \cdot \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \right\rangle - \langle \tilde{\mathbf{F}}' \cdot \tilde{\boldsymbol{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle \right\}. \tag{A.10}$$

By the duality theorem (cf. Zeidler [16], Chapter 2.12, Theorem 2.F) we know

PROPOSITION A.2. *If* $(\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j})$ *is a saddle point of* $\mathcal{S}_{ij}$, $\tilde{\mathbf{E}}_{i,j}$ *is a minimizer of* (A.8) *and* $\tilde{\mathbf{E}}'_{i,j}$ *is a maximizer of* (A.10). *The converse holds true provided that*

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) = K(\boldsymbol{e}_i, \boldsymbol{e}_j). \tag{A.11}$$

It is straightforward to check that the functionals in (A.10) and (A.8) are *strictly convex* in $\check{\mathcal{V}}_g(\tilde{\mathbf{H}})$ and so the pair of minimizer and maximizer is unique up to a constant. Thus we have

PROPOSITION A.3. *The saddle point* $(\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j})$ *of* $\mathcal{S}_{ij}$, *and so the minimizer* (*maximizer*) *of* (A.8)((A.10)), *exists and is unique up to constant.*

The uniqueness of the saddle point can also be shown as follows. The necessary condition for the minimizer of (A.8) is the Euler–Lagrange equation of (A.8):

$$\langle \tilde{\Gamma}\tilde{\mathbf{E}}_{i,j} \cdot \tilde{\Gamma}\tilde{\mathbf{F}} \rangle + \langle \tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\Gamma}\tilde{\mathbf{E}}_{i,j} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\Gamma}\tilde{\mathbf{F}} \rangle + \left\langle \tilde{\Gamma}\tilde{\mathbf{H}} \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\Gamma}\tilde{\mathbf{F}} \right\rangle$$

$$+ \left\langle \tilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\mathbf{H}} \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right\rangle = 0 \tag{A.12}$$

for all $\tilde{\mathbf{F}} \in \check{\mathcal{V}}_g(\tilde{\mathbf{H}})$. By the Riesz representation theorem applied to $\check{\mathcal{V}}_g(\tilde{\mathbf{H}})$, the minimizer $\tilde{\mathbf{E}}_{i,j}$ exists and is the unique, up to a constant.

Similarly, the maximizer $\tilde{\mathbf{E}}'_{i,j}$ of (A.10) is the unique, up to a constant, solution of the Euler–Lagrange equation of (A.10):

$$\langle \tilde{\Gamma}\tilde{\mathbf{E}}'_{i,j} \cdot \tilde{\Gamma}\tilde{\mathbf{F}}' \rangle + \langle \tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\Gamma}\tilde{\mathbf{E}}'_{i,j} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\Gamma}\tilde{\mathbf{F}}' \rangle + \left\langle \tilde{\Gamma}\tilde{\mathbf{H}} \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}\tilde{\Gamma}\tilde{\mathbf{F}}' \right\rangle$$

$$+ \left\langle \tilde{\mathbf{F}}' \cdot \tilde{\Gamma}\tilde{\mathbf{H}} \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right\rangle = 0 \tag{A.13}$$

for all $\tilde{\mathbf{F}}' \in \check{\mathcal{V}}_g(\tilde{\mathbf{H}})$.

It is easy to see now that

$$\sup_{\tilde{\mathbf{F}}' \in \mathscr{V}_g(\tilde{\mathbf{H}})} \mathscr{S}_{ij}(\tilde{\mathbf{E}}_{i,\,j}, \tilde{\mathbf{F}}') = \mathscr{S}_{ij}(\tilde{\mathbf{E}}_{i,\,j}, \tilde{\mathbf{E}}'_{i,\,j}) = \inf_{\tilde{\mathbf{F}} \in \mathscr{V}_g(\tilde{\mathbf{H}})} \mathscr{S}_{ij}(\tilde{\mathbf{F}}, \tilde{\mathbf{E}}'_{i,\,j}) \tag{A.14}$$

and thus the following Euler–Lagrange equations (cf. (A.7), (A.9)) hold

$$\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_{i,j} = -\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_{i,j} - \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \tag{A.15}$$

$$\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_{i,j} = -\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_{i,j} - \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \tag{A.16}$$

(A.15) and (A.16) are understood in the weak sense. Adding and subtracting (A.15) and (A.16) we have

$$\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i = -\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i - \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i \tag{A.17}$$

$$\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_j = \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_j + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_j \tag{A.18}$$

where

$$\tilde{\mathbf{E}}_i := \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_{i,j} + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_{i,j} \tag{A.19}$$

$$\tilde{\mathbf{E}}'_j := \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_{i,j} - \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_{i,j}. \tag{A.20}$$

Equations (A.17) and (A.18) are precisely the cell problem and its adjoint. Note that $\tilde{\mathbf{E}}i, \tilde{\mathbf{E}}'_j \in \mathscr{V}_g(\tilde{\mathbf{H}})$ because $\tilde{\mathbf{E}}_{i,\,j}, \tilde{\mathbf{E}}'_{i,\,j} \in \mathscr{V}_g(\tilde{\mathbf{H}})$. Thus on the solution space $\mathscr{V}_g(\tilde{\mathbf{H}})$, the Euler–Lagrange Eqs. (A.15), (A.16) are equivalent to the cell problem and its adjoint via (A.19), (A.20), or, equivalently,

$$\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_{ij} = \frac{\tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}'_j}{2} \tag{A.21}$$

$$\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}'_{ij} = \frac{\tilde{\mathbf{E}}_i - \tilde{\mathbf{E}}'_j}{2}. \tag{A.22}$$

The existence and uniqueness of the saddle point, up to a constant, imply the existence and uniqueness of the cell problem and its adjoint if the mean fields are specified

$$\langle \tilde{\mathbf{E}}_i \rangle = 0 \tag{A.23}$$

$$\langle \tilde{\mathbf{E}}'_j \rangle = 0. \tag{A.24}$$

A.2. *Two Identities for the Effective Diffusivity*

It is now straightforward to check that $\delta_{ij} + J(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sigma^*_{i,j}$.

In terms of (A.21) and (A.22), $J(\boldsymbol{e}_i, \boldsymbol{e}_j)$ can be written as

$$J(\boldsymbol{e}_i,\boldsymbol{e}_j) = \mathscr{S}(\tilde{\mathbf{E}}_{i,j}, \tilde{\mathbf{E}}'_{i,j}) \tag{A.25}$$

$$= \langle (\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i) \cdot \tilde{\mathbf{E}}'_j \rangle - \langle \tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_j \rangle \tag{A.26}$$

$$= \langle \tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \boldsymbol{e}_j \rangle \tag{A.27}$$

$$= \langle \tilde{\mathbf{H}}\tilde{\mathbf{E}}_i \cdot \boldsymbol{e}_j \rangle \tag{A.28}$$

The weak form of Eq. (A.17) is used in the derivation. The last expression plus δ_{ij} yields $\sigma^*_{i,j}$ by the Definition (4.3).

In general the effective diffusivity matrix σ^* is not symmetric but only the symmetric part appears in the homogenized equation.

Let us derive another useful identity for $\sigma^*_{i,j}$

$$\tfrac{1}{2}(\sigma^*_{i,j} + \sigma^*_{j,i}) = 1 + \langle \tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{E}}_j \rangle \tag{A.29}$$

if the mean filed (A.23) is satisfied.

For the diagonal entries where $i = j$, reversing the derivation in (A.26)–(A.28) we have

$$\sigma^*_{i,i} = 1 + \langle \tilde{\mathbf{H}}\tilde{\mathbf{E}}_i \cdot \boldsymbol{e}_i \rangle \tag{A.30}$$

$$= 1 + \langle (\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i) \cdot \tilde{\mathbf{E}}'_i \rangle - \langle \tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i \rangle \tag{A.31}$$

$$= 1 + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{E}}_i \rangle + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{E}}_i \rangle \tag{A.32}$$

$$= 1 + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \rangle \tag{A.33}$$

For the off-diagonal entries where $i \neq j$, reversing the derivation in (A.26) we have

$$\sigma^*_{i,j} = \langle \tilde{\mathbf{H}}\tilde{\mathbf{E}}_i \cdot \boldsymbol{e}_j \rangle \tag{A.34}$$

$$= \langle (\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i) \cdot \tilde{\mathbf{E}}_j \rangle - \langle \tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_j \rangle \tag{A.35}$$

$$= \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{E}}_j \rangle + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{E}}_j \rangle + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i \cdot \tilde{\mathbf{E}}_j \rangle - \langle \tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_j \rangle. \tag{A.36}$$

Hence,

$$\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_j \rangle = \sigma^*_{i,j} - \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{E}}_j \rangle - \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i \cdot \tilde{\mathbf{E}}_j \rangle + \langle \tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_j \rangle. \tag{A.37}$$

But the expression on the left side of (A.37) is symmetrical with respect to i and j, so we have also

$$\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_i \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_j \rangle = \sigma^*_{j,i} - \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{E}}_j \cdot \tilde{\mathbf{E}}_i \rangle - \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_j \cdot \tilde{\mathbf{E}}_i \rangle + \langle \tilde{\mathbf{E}}_j \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\boldsymbol{e}_i \rangle \tag{A.38}$$

by interchanging the indexes. The identity (A.29) follows by adding (A.37) and (A.38), and using the skew-adjointness of the operator $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}$.

A.3. *Cut-Off and Convergence*

Since the space $\mathscr{V}_g(\tilde{\mathbf{H}})$ is the domain of the graph closure of the operator $\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{\Gamma}}$ on the domain

$$\mathscr{D}_g = \{\mathbf{G} \in \mathscr{V}_g;\, \tilde{\mathbf{H}}\mathbf{G} \in \mathscr{V}\} \tag{A.39}$$

the variations in (A.2), (A.47) and (A.10) can be restricted to $\mathscr{D}_g$.

Note that for any $\tilde{\mathbf{F}} \in \mathscr{D}_g$ we have

$$\langle [(\tilde{\mathbf{H}} - \tilde{\mathbf{H}}^{(M)})\tilde{\mathbf{F}}]^2 \rangle \to 0. \tag{A.40}$$

as $M \to \infty$. Consequently,

$$\langle \tilde{\mathbf{\Gamma}}(\tilde{\mathbf{H}} - \tilde{\mathbf{H}}^{(M)})\,\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}(\tilde{\mathbf{H}} - \tilde{\mathbf{H}}^{(M)})\,\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \rangle \to 0 \tag{A.41}$$

as $M \to \infty$. Here the stream matrix $\tilde{\mathbf{H}}^{(M)}$ is the level M truncation of $\tilde{\mathbf{H}}$.

Let us consider the analogous variational problem with the M-level truncated stream matrix $\tilde{\mathbf{H}}^{(M)}$, defined in Section 2,

$$J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) := \inf_{\tilde{\mathbf{F}} \in \mathscr{V}_g} \sup_{\tilde{\mathbf{F}}' \in \mathscr{V}_g} \mathscr{S}_M(\tilde{\mathbf{F}}, \tilde{\mathbf{F}}'), \tag{A.42}$$

where

$$\begin{aligned} \mathscr{S}_M(\tilde{\mathbf{F}}, \tilde{\mathbf{F}}') := {} & \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \rangle - 2\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}' \rangle - \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \rangle \\ & + \langle \tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle - \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i + \boldsymbol{e}_j) \cdot \tilde{\mathbf{F}}' \rangle \end{aligned} \tag{A.43}$$

for any $\tilde{\mathbf{F}}, \tilde{\mathbf{F}}' \in \mathscr{V}_g$. Note that $\mathscr{V}_g(\tilde{\mathbf{H}}^{(M)}) = \mathscr{V}_g$ because of the boundedness of $\tilde{\mathbf{H}}^{(M)}$. $\mathscr{S}_M(\tilde{\mathbf{F}}, \tilde{\mathbf{F}}')$ can also be simply written as

$$\mathscr{S}_M(\tilde{\mathbf{F}}, \tilde{\mathbf{F}}') := \langle \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}} \rangle - 2\langle \tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}}' \rangle - \langle \tilde{\mathbf{F}}' \cdot \tilde{\mathbf{F}}' \rangle \tag{A.44}$$

with

$$\langle \tilde{\mathbf{F}} \rangle = \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2}, \qquad \langle \tilde{\mathbf{F}}' \rangle = \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2}. \tag{A.45}$$

By Proposition A.1,

$$J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) := \sup_{\substack{\tilde{\mathbf{F}}' \in \mathscr{V}_g \\ \langle \tilde{\mathbf{F}}' \rangle = (\boldsymbol{e}_i - \boldsymbol{e}_j)/2}} \inf_{\substack{\tilde{\mathbf{F}} \in \mathscr{V}_g \\ \langle \tilde{\mathbf{F}} \rangle = (\boldsymbol{e}_i + \boldsymbol{e}_j)/2}} \mathscr{S}_M(\tilde{\mathbf{F}}, \tilde{\mathbf{F}}') \tag{A.46}$$

The same procedure leading to (A.8), (A.10) now gives

$$J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) = \inf_{\tilde{\mathbf{F}} \in \mathscr{V}_g} \Bigg\{ \langle \tilde{\Gamma}\tilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\mathbf{F}} \rangle + \Bigg\langle \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\Gamma}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \cdot \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\Gamma}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \Bigg\rangle + \langle \tilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle \Bigg\}$$

$$= \inf_{\substack{\mathbf{F} \in \mathscr{V}_g \\ \langle \tilde{\mathbf{F}} \rangle = (\boldsymbol{e}_i + \boldsymbol{e}_j)/2}} \Bigg\{ \langle \tilde{\mathbf{F}} \cdot \tilde{\mathbf{F}} \rangle + \langle \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}} \rangle - \langle \tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}} \cdot (\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle - \left| \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right|^2 \Bigg\} \tag{A.47}$$

and

$$J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sup_{\tilde{\mathbf{F}}' \in \mathscr{V}_g} \Bigg\{ -\langle \tilde{\Gamma}\tilde{\mathbf{F}}' \cdot \tilde{\Gamma}\tilde{\mathbf{F}}' \rangle - \Bigg\langle \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\Gamma}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \cdot \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\Gamma}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \Bigg\rangle - \langle \tilde{\mathbf{F}}' \cdot \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle \Bigg\}$$

$$= \sup_{\substack{\tilde{\mathbf{F}}' \in \mathscr{V}_g \\ \langle \tilde{\mathbf{F}}' \rangle = (\boldsymbol{e}_i - \boldsymbol{e}_j)/2}} \Bigg\{ -\langle \tilde{\mathbf{F}}' \cdot \tilde{\mathbf{F}}' \rangle - \langle \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}}' \cdot \tilde{\Gamma}\tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}}' \rangle + \langle \tilde{\mathbf{H}}^{(M)}\tilde{\mathbf{F}}' \cdot (\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle + \left| \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right|^2 \Bigg\}. \tag{A.48}$$

Next we prove the convergence lemma:

Lemma A.1.

$$\lim_{M \to \infty} J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) = J(\boldsymbol{e}_i, \boldsymbol{e}_j), \qquad \forall i, j. \tag{A.49}$$

Proof. We first show the upper bound: $\limsup_{M \to \infty} J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) \leqslant J(\boldsymbol{e}_i, \boldsymbol{e}_j)$, $\forall i, j$, using the minimum principles (A.47) and (A.8).

In view of (A.8), for given $\delta > 0$ there exists a $\tilde{\mathbf{F}} \in \mathscr{D}_g$ such that

$$\langle \tilde{\Gamma}\tilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\mathbf{F}} \rangle + \Bigg\langle \tilde{\Gamma}\tilde{\mathbf{H}} \left(\tilde{\Gamma}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \cdot \tilde{\Gamma}\tilde{\mathbf{H}} \left(\tilde{\Gamma}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \Bigg\rangle + \langle \tilde{\mathbf{F}} \cdot \tilde{\Gamma}\tilde{\mathbf{H}}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle \leqslant J(\boldsymbol{e}_i, \boldsymbol{e}_j) + \delta. \tag{A.50}$$

For the same $\tilde{\mathbf{F}}$, and sufficiently large M the left-side of (A.50) is bigger than

$$\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \rangle + \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \right\rangle + \langle \tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle - \delta \tag{A.51}$$

which in turn is bigger than

$$J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) - \delta \tag{A.52}$$

in view of the minimum principle (A.47). Thus we have that

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) \geqslant J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) + 2\delta \tag{A.53}$$

for sufficiently large M. This proves the upper bound.

We turn to the lower bound: $\liminf_{M \to \infty} J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) \geqslant J(\boldsymbol{e}_i, \boldsymbol{e}_j)$, $\forall i, j$.

By the maximum principle (A.10), there exists $\tilde{\mathbf{F}}' \in \mathscr{D}_g$ for given $\delta > 0$, such that

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) - \delta \leqslant -\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \rangle - \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \right\rangle - \langle \tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle.$$

The right side of (A.54) is bounded by

$$-\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \rangle - \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \right\rangle - \langle \tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle + \delta \tag{A.54}$$

for sufficiently large M, which, in turn, is bounded by

$$J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) + \delta \tag{A.55}$$

by the maximum principle (A.48). Thus, we have

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) - 2\delta \leqslant \liminf_{M \to \infty} J_M(\boldsymbol{e}_i, \boldsymbol{e}_j) \tag{A.56}$$

for any $\delta > 0$. This proves the lower bound. ∎

We now show that the trial functions for the variational principles (A.47) and (A.47) can be restricted to bounded gradient fields. The proof essentially follows from Lemma A.1 and the density of $\mathscr{B}_g$ in $\mathscr{V}_g$.

LEMMA A.2. *The minimum* (A.8) (*maximum in* (A.10)) *is achieved in the space of bounded gradient fields* $\mathscr{B}_g$.

Proof. For the minimum principle, suffice it to show that given $\delta > 0$, there exists bounded gradient field $\tilde{\mathbf{F}}$ with $\langle \tilde{\mathbf{F}} \rangle = (\boldsymbol{e}_i + \boldsymbol{e}_j)/2$ such that

$$\sigma^*_{i,j} + \delta \geqslant \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \rangle + \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \right\rangle$$
$$+ \langle \tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle. \tag{A.57}$$

By Lemma A.1, we have that

$$\sigma^*_{i,j} + \frac{\delta}{2} \geqslant \sigma^{(M)}_{i,j} \tag{A.58}$$

for sufficiently large M. By the remark in the beginning of the section, there exists $\tilde{\mathbf{F}} \in \mathscr{B}_g$ such that

$$\sigma^*_{i,j} - \delta \geqslant \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \rangle + \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} + \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right) \right\rangle$$
$$+ \langle \hat{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle. \tag{A.59}$$

Moreover, (A.59) is valid uniformly in M by (A.40) and (A.41). Equations (A.58), (A.59) together with (A.40), (A.41) imply (A.57).

We turn to the maximum principle. Suffice it to show that given $\delta > 0$, there exists $\tilde{\mathbf{F}}' \in \mathscr{B}_g$ with $\langle \tilde{\mathbf{F}}' \rangle = (\boldsymbol{e}_i - \boldsymbol{e}_j)/2$ such that

$$\sigma^*_{i,j} - \delta \leqslant -\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \rangle - \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \right\rangle$$
$$- \langle \tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle. \tag{A.60}$$

By Lemma A.1, we have that

$$\sigma^*_{i,j} - \frac{\delta}{2} \leqslant \sigma^{(M)}_{i,j} \tag{A.61}$$

for sufficiently large M. Thus it follows from (A.48) and the density of $\mathscr{B}_g$ for bounded $\tilde{\mathbf{H}}^{(M)}$ that there exists $\mathbf{F}' \in \mathscr{B}_g$ such that

$$\sigma^*_{i,j} - \delta \leqslant -\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' \rangle - \left\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \right.$$
$$\left. \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)} \left(\tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}' + \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right) \right\rangle - \langle \tilde{\mathbf{F}}' \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}^{(M)}(\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle. \tag{A.62}$$

Note that (A.62) is valid for all sufficiently large M due to (A.40) and (A.41). Thus, (A.60) follows by passing to the limit $M \to \infty$. ▮

Note that the space of *bounded, continuous* gradient fields $\mathscr{C}_g$ is dense in $\mathscr{V}_g$. By the same argument, we have

LEMMA A.3. *The minimum in* (A.8) (*maximum in* (A.10)) *is achieved in the space of bounded continuous gradient fields* $\mathscr{C}_g$.

Let us turn to the proof of Lemma 4.1.

A.4. *Approximate Correctors*: *Proof of Lemma* 4.1

Writing a general trial field $\tilde{\mathbf{G}}$ in the form

$$\tilde{\mathbf{G}} = \tilde{\mathbf{E}}_{i,\,i} + \tilde{\mathbf{F}} \tag{A.63}$$

and substituting it in the functional in (A.8) we have by straightforward calculation using (A.12) that

$$J(\boldsymbol{e}_i, \boldsymbol{e}_j) + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}} \rangle + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{F}} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{F}} \rangle. \tag{A.64}$$

By Lemma A.3, for given $\delta > 0$ there exists $\tilde{\mathbf{G}}_{i,\,i} \in \mathscr{C}_g$ such that

$$\langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}_{i,\,i} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{F}}_{i,\,i} \rangle + \langle \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{F}}_{i,\,i} \cdot \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{F}}_{i,\,i} \rangle \leqslant \delta \tag{A.65}$$

where

$$\tilde{\mathbf{F}}_{i,\,i} := \tilde{\mathbf{G}}_{i,\,i} - \tilde{\mathbf{E}}_{i,\,i}. \tag{A.66}$$

Or, equivalently, there exist $\tilde{\mathbf{G}}_{i,\,i}$ in the space $\mathscr{B}_g$ such that $\langle \tilde{\mathbf{G}}_{i,\,i} \rangle = 0$ and

$$\langle |\tilde{\mathbf{G}}_{i,\,i} + \boldsymbol{e}_i - \tilde{\mathbf{E}}_{i,\,i}|^2 \rangle < \delta^2/2 \tag{A.67}$$

$$\langle |\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}_{i,\,i} + \boldsymbol{e}_i - \tilde{\mathbf{E}}_{i,\,i})|^2 \rangle < \delta^2/2 \tag{A.68}$$

Likewise, there exist $\tilde{\mathbf{G}}'_{i,\,i} \in \mathscr{C}_g$ such that $\langle \tilde{\mathbf{G}}_{i,\,i} \rangle = 0$ and

$$\langle |\tilde{\mathbf{G}}'_{i,\,i} + \boldsymbol{e}_i - \tilde{\mathbf{E}}'_{i,\,i}|^2 \rangle < \delta^2/2 \tag{A.69}$$

$$\langle |\tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}'_{i,\,i} + \boldsymbol{e}_i - \tilde{\mathbf{E}}'_{i,\,i})|^2 \rangle < \delta^2/2. \tag{A.70}$$

Thus it follows from (A.67), (A.70), (A.15) and (A.16) that

$$\langle |\tilde{\mathbf{G}}_{i,\,i} + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}\tilde{\mathbf{G}}'_{i,\,i}|^2 \rangle < \delta^2. \tag{A.71}$$

Similarly,

$$\langle |\tilde{\mathbf{G}}'_{i,\,i} + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}_{i,\,i} + \boldsymbol{e}_i)|^2 \rangle < \delta^2. \tag{A.72}$$

follows from (A.68), (A.69), (A.15), and (A.16).

Hence for the sum

$$\tilde{\mathbf{G}}_i := \tilde{\mathbf{G}}_{i,i} + \tilde{\mathbf{G}}'_{i,i} \tag{A.73}$$

we have

$$\langle |\tilde{\mathbf{G}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i)|^2 \rangle < \delta^2 \tag{A.74}$$

which is inequality (4.15).

The proof of (4.16) follows from the similar argument and the version of Lemma A.3 for the adjoint cell problem (i.e. with $\tilde{\mathbf{H}}$ replaced by $-\tilde{\mathbf{H}}$).

To show inequality (4.17) let us consider the following identities:

$$\begin{aligned}
\sigma^*_{i,j} &= \langle (\mathbf{I} - \tilde{\mathbf{H}})(\tilde{\mathbf{E}}'_j + \boldsymbol{e}_j) \cdot \boldsymbol{e}_i \rangle \\
&= \langle (\mathbf{I} - \tilde{\mathbf{H}})(\tilde{\mathbf{E}}'_j + \boldsymbol{e}_j) \cdot (\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \rangle \\
&= \langle (\tilde{\mathbf{E}}'_j + \boldsymbol{e}_j) \cdot (\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \rangle \\
&= \langle (\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \rangle + \langle (\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \cdot \tilde{\mathbf{E}}'_j \rangle.
\end{aligned} \tag{A.75}$$

This leads to the identity

$$\begin{aligned}
\langle (\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \rangle - \sigma^*_{i,j} &= -\langle (\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \cdot \tilde{\mathbf{E}}'_j \rangle \\
&= -\langle \tilde{\mathbf{\Gamma}}(\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \cdot \tilde{\mathbf{E}}'_j \rangle \\
&= -\langle (\tilde{\mathbf{G}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i)) \cdot \tilde{\mathbf{E}}'_j \rangle.
\end{aligned} \tag{A.76}$$

Thus we have by the Cauchy–Schwarz inequality that

$$\begin{aligned}
|\langle (\mathbf{I} + \tilde{\mathbf{H}})(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \rangle - \sigma^*_{i,j}| &\leqslant \sqrt{\langle \tilde{\mathbf{E}}'_j \cdot \tilde{\mathbf{E}}'_j \rangle} \sqrt{\langle |\tilde{\mathbf{G}}_i + \tilde{\mathbf{\Gamma}}\tilde{\mathbf{H}}(\tilde{\mathbf{G}}_i + \boldsymbol{e}_i)|^2 \rangle} \\
&\leqslant \delta \sqrt{\langle \tilde{\mathbf{E}}'_j \cdot \tilde{\mathbf{E}}'_j \rangle}
\end{aligned} \tag{A.77}$$

which is the desired inequality (4.17). The inequality (4.18) can be similarly proved. ∎

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