PERGAMON

International Journal of Solids and Structures 38 (2001) 2989–3005

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The crack problem for nonhomogeneous materials under antiplane shear loading — a displacement based formulation

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Received 30 August 1999; in revised form 11 May 2000

Abstract

This paper presents a displacement based integral equation formulation for the mode III crack problem in a nonhomogeneous medium with a continuously differentiable shear modulus, which is assumed to be an exponential function, i.e., $G(x) = G_0 \exp(\beta x)$. This formulation leads naturally to a hypersingular integral equation. The problem is solved for a finite crack and results are given for crack profiles and stress intensity factors. The results are affected by the parameter β describing the material nonhomogeneity. This study is motivated by crack problems in strain-gradient elasticity theories where higher order singular integral equations naturally arise even in the slope-based formulation. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Functionally graded materials; Mode III crack; Integral equation method; Fredholm integral equation; Hypersingular integrals; Asymptotic analysis; Collocation method; Stress intensity factors

1. Introduction

Recently, the study of nonhomogeneous solids has gained renewed importance with the advances in the field of functionally graded materials (FGMs) – see, for example the proceedings of the FGM'98 Conference held in Dresden, Germany (Kaysser, 1999). From the point view of continuum mechanics, such materials can be treated as nonhomogeneous solids modeled by variable elasticity moduli. This is the viewpoint taken in the present work.

Erdogan (1985) has solved the mode III crack problem for bonded nonhomogeneous materials using a slope-based formulation. In the present paper, a displacement-based formulation is derived and a comparison of approaches is provided. Emphasis is placed on selection and interpolation of the primary density function using Chebyshev polynomials of both the first and the second kinds.

The goal for the remainder of this paper is to develop a comprehensive presentation with focus on integral equation methods for fracture problems. Thus, the next sections are organized as follows. First, the

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PII: S0020-7683(00)00217-1

governing equation and corresponding boundary conditions for the mode III crack problem are given. Next, the Fourier analysis method is addressed which includes the decomposition of the kernel in order to separate the nonsingular part from the singular part. Subsequently, the resulting hypersingular integral equation is derived. Afterwards, a comparison between displacement versus slope formulations is given. Next, the discretization of the governing integral equation is provided and stress intensity factors (SIFs) are derived from first principles. Finally, conclusions are inferred and potential extensions of this work are pointed out. Three brief, but relevant, appendices supplement the present work.

2. Governing differential equations and boundary conditions

Consider a mode III crack problem in a nonhomogeneous elastic medium with the shear modulus variation illustrated in Fig. 1. The nontrivial equilibrium condition is

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0. \tag{1}$$

If the shear modulus G is a function of x and takes the exponential form

$$G(x) = G_0 e^{\beta x}, \tag{2}$$

where G_0 and β are material constants, then the equilibrium condition (1) can be rewritten as a partial differential equation (PDE) in terms of the z component of the displacement vector, $w(x,y)$, i.e.,

$$\nabla^2 w(x,y) + \beta \frac{\partial w(x,y)}{\partial x} = 0. \tag{3}$$

An interesting observation is that as $\beta \to 0$, the above PDE is reduced to the classical harmonic equation. Thus this antiplane shear problem for nonhomogeneous material with shear modulus $G(x) = G_0 e^{\beta x}$ can be considered as a perturbation of the classical antiplane shear problem for homogeneous material.

Furthermore, the governing PDE (3) is solved under the following mixed boundary conditions:

$$\begin{aligned} &w(x,0) = 0, \quad x \notin [c,d], \\ &\sigma_{yz}(x,0^+) = p(x), \quad x \in (c,d), \end{aligned} \tag{4}$$

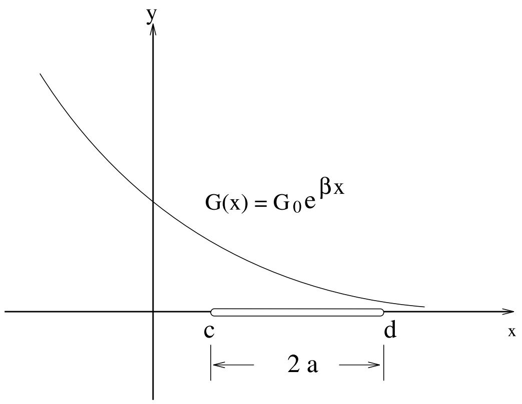

Fig. 1. Antiplane shear problem for a nonhomogeneous material with shear modulus $G(x) = G_0 e^{\beta x}$.

where $p(x)$ is the traction function along the crack surfaces (c, d). Due to symmetry, one can only consider the upper half plane $y > 0$ for this problem. Thus, the governing differential equation and boundary conditions can be summarized by

$$\begin{aligned} &\nabla^2 w(x,y) + \beta \tfrac{\partial w(x,y)}{\partial x} = 0, \quad -\infty < x < \infty, \quad y > 0, \\ &w(x,0) = 0, \quad x \notin [c,d], \\ &\sigma_{yz}(x,0^+) = p(x), \quad x \in (c,d). \end{aligned} \tag{5}$$

3. Fourier transform analysis

Let $W(\xi, y)$ be the Fourier transform of $w(x,y)$ defined by

$$\mathscr{F}\{w\}(\xi,y) = W(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x,y) \mathrm{e}^{\mathrm{i}x\xi}\, \mathrm{d}x, \tag{6}$$

so that $w(x,y)$ is the inverse Fourier transform of the function $W(\xi, y)$, i.e.,

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(\xi,y) \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}\xi. \tag{7}$$

With this approach, one transforms the PDE (3) for $w(x,y)$ into an ordinary differential equation (ODE) for $W(\xi, y)$, i.e.,

$$\frac{\partial^2 W(\xi,y)}{\partial y^2} - \xi^2 - \mathrm{i}\beta\xi = 0. \tag{8}$$

The corresponding characteristic equation to the ODE (8) above is

$$[\lambda(\xi)]^2 = \xi^2 + \mathrm{i}\beta\xi. \tag{9}$$

To satisfy the far field boundary condition, $\lim_{y\to\infty} w(x,y) = 0$, we choose the root $\lambda(\xi)$ with nonpositive real part,

$$\lambda(\xi) = \frac{-1}{\sqrt{2}} \sqrt{\sqrt{\xi^4 + \beta^2\xi^2} + \xi^2} - \frac{\mathrm{i}}{\sqrt{2}} \mathrm{sgn}(\beta\xi) \sqrt{\sqrt{\xi^4 + \beta^2\xi^2} - \xi^2}\ , \tag{10}$$

where the signum function $\mathrm{sgn}(\cdot)$ is defined as

$$\mathrm{sgn}(\eta) = \begin{cases} 1, & \eta > 0 \\ 0, & \eta = 0 \\ -1, & \eta < 0. \end{cases} \tag{11}$$

Thus, $W(\xi, y)$ is found to be

$$W(\xi,y) = A(\xi) \mathrm{e}^{\lambda(\xi)y}, \tag{12}$$

and, by Eq. (7), $w(x,y)$ can be expressed as

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(\xi)\mathrm{e}^{\lambda(\xi)y}\right] \mathrm{e}^{-\mathrm{i}x\xi}\, \mathrm{d}\xi, \tag{13}$$

where $A(\xi)$ is to be determined by the boundary conditions in (4).

As the limit of $y \to 0^+$ is taken,

$$w(x,0^+) = \lim_{y\to 0^+} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(\xi)e^{\lambda(\xi)y}\right] e^{-ix\xi}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\xi) e^{-ix\xi}\, d\xi, \tag{14}$$

that is, $w(x,0^+)$ is the inverse Fourier transform of $A(\xi)$. By inverting the Fourier transform, one obtains

$$A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(x,0^+) e^{ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \int_{c}^{d} w(t,0^+) e^{it\xi}\, dt, \tag{15}$$

where the first boundary condition in Eq. (4) and a change of dummy variable ($x \leftrightarrow t$) have been applied.

On the other hand, the stress σ_{yz} is given by

$$\sigma_{yz}(x,y) = G(x)\frac{\partial w(x,y)}{\partial y} = \frac{G(x)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(\xi)\lambda(\xi)e^{\lambda(\xi)y}\right] e^{-ix\xi}\, d\xi. \tag{16}$$

Replacing $A(\xi)$ in Eq. (16) above by the expression in Eq. (15), one gets

$$\begin{aligned}\sigma_{yz}(x,y) &= \frac{G(x)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{c}^{d} w(t,0^+) e^{it\xi}\, dt\right] \lambda(\xi) e^{\lambda(\xi)y} e^{-ix\xi}\, d\xi \\ &= \frac{G(x)}{2\pi} \int_{c}^{d} w(t,0^+) \int_{-\infty}^{\infty} \lambda(\xi) e^{\lambda(\xi)y} e^{i(t-x)\xi}\, d\xi\, dt. \end{aligned} \tag{17}$$

Defining

$$K(\xi,y) = \lambda(\xi)e^{\lambda(\xi)y}, \tag{18}$$

and using the second boundary condition in Eq. (4), one reaches

$$\begin{aligned}\sigma_{yz}(x,0^+) &= \lim_{y\to 0^+} \sigma_{yz}(x,y), \quad c < x < d \\ &= \lim_{y\to 0^+} \frac{G(x)}{2\pi} \int_{c}^{d} w(t,0^+) \int_{-\infty}^{\infty} K(\xi,y) e^{i(t-x)\xi}\, d\xi\, dt \\ &= \frac{G(x)}{\pi} \int_{c}^{d} w(t,0^+)\, \mathrm{kernel}\,(x-t)\, dt \\ &= p(x) \end{aligned} \tag{19}$$

with

$$2\,\mathrm{kernel}\,(x-t) = \lim_{y\to 0^+} \int_{-\infty}^{\infty} K(\xi,y) e^{i(t-x)\xi}\, d\xi, \tag{20}$$

where Eq. (19) and the above kernel are interpreted in a limit sense.

4. Asymptotic analysis and kernel decomposition

In order to make the kernel explicit (Eq. (20)) and separate its singular and regular parts; we need to investigate the asymptotic behavior, $|\xi| \to +\infty$, of

$$K(\xi) \equiv K(\xi,0^+) = \lambda(\xi). \tag{21}$$

A simple asymptotic analysis gives the following results:

$$\mathscr{R}(\lambda(\xi)) = \frac{-1}{\sqrt{2}}\sqrt{\sqrt{\xi^4 + \beta^2\xi^2} + \xi^2} \overset{|\xi|\to\infty}{\sim} -|\xi|, \tag{22}$$

$$\mathrm{i}\mathscr{I}(\lambda(\xi)) = \frac{-\mathrm{i}}{\sqrt{2}}\,\mathrm{sgn}\,(\beta\xi)\sqrt{\sqrt{\xi^4+\beta\xi^2}-\xi^2} \overset{|\xi|\to\infty}{\sim} -\frac{\mathrm{i}\beta}{2}\frac{|\xi|}{\xi}, \tag{23}$$

where $\mathscr{R}(\cdot)$ and $\mathscr{I}(\cdot)$ denote the real part and the imaginary part of the argument, respectively. Thus, we have the decomposition

$$K(\xi) = [K(\xi) - K_\infty(\xi)] + K_\infty(\xi), \tag{24}$$

with closed form expressions given by

$$K_\infty(\xi) = -|\xi| - \frac{\mathrm{i}\beta}{2}\frac{|\xi|}{\xi}, \tag{25}$$

$$\begin{aligned} K(\xi) - K_\infty(\xi) = {} & \frac{-\beta^2\sqrt{|\xi|}}{2\left(\sqrt{|\xi|}+\frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^2+\beta^2}+|\xi|}\right)\left(|\xi|+\sqrt{\xi^2+\beta^2}\right)} \\ & + \frac{\mathrm{i}\beta^4|\xi|/\xi}{2\left(\beta+\sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2}\right)\left(2\xi^2+\beta^2+2\sqrt{\xi^4+\beta^2\xi^2}\right)}, \end{aligned} \tag{26}$$

after a bit of algebraic manipulation (Appendix A). In general, the function $K(\xi)$ may be complicated (Konda and Erdogan, 1994) and so is the corresponding asymptotic analysis. In any case, the singular part, such as Eq. (25), can be separated by using a symbolic calculation software such as MAPLE and the regular kernel can be dealt with numerically.

Thus, Eq. (20) becomes

$$\begin{aligned} 2\,\mathrm{kernel}\,(x,t) &= \lim_{y\to 0^+}\int_{-\infty}^{\infty} K(\xi,y)\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi \\ &= \underbrace{\lim_{y\to 0^+}\int_{-\infty}^{\infty}[K(\xi,y)-K_\infty(\xi,y)]\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi}_{\text{nonsingular part}} + \underbrace{\lim_{y\to 0^+}\int_{-\infty}^{\infty}K_\infty(\xi,y)\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi}_{\text{singular part}}, \end{aligned} \tag{27}$$

where

$$K_\infty(\xi,y) = \left(-|\xi| - \frac{\mathrm{i}\beta}{2}\frac{|\xi|}{\xi}\right)\mathrm{e}^{-|\xi|y}. \tag{28}$$

Therefore, the decomposition of the kernel(x,t) (Eq. (20)) into nonsingular and singular parts has been achieved.

5. Hypersingular integral equation

The singular part in Eq. (27) can be shown to converge in the sense of distribution (see Appendix B and Fannjiang et al., 2000):

$$\lim_{y\to 0^+}\int_{-\infty}^{\infty}K_\infty(\xi,y)\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi = \lim_{y\to 0^+}\int_{-\infty}^{\infty}\left(-|\xi|-\frac{\mathrm{i}\beta}{2}\frac{|\xi|}{\xi}\right)\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi = \frac{2}{(t-x)^2}+\frac{\beta}{t-x}. \tag{29}$$

The expression on the right side of Eq. (29) is a distribution (or generalized function) defined via Hadamard's finite-part integral (including Cauchy principal value). The nonsingular part in Eq. (27) can be obtained by means of Eq. (26), which leads to

$$N(x,t) = \frac{1}{2}\lim_{y\to 0^+}\int_{-\infty}^{\infty}[K(\xi,y)-K_\infty(\xi,y)]\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi = \frac{1}{2}\int_{-\infty}^{\infty}[K(\xi)-K_\infty(\xi)]\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi$$
$$= -\int_0^\infty \frac{\beta^2\sqrt{\xi}\cos[(t-x)\xi]}{\left(2\sqrt{\xi}+\sqrt{2}\sqrt{\sqrt{\xi^2+\beta^2}+\xi}\right)\left(\xi+\sqrt{\xi^2+\beta^2}\right)}\,\mathrm{d}\xi$$
$$-\int_0^\infty \frac{\beta^4\sin[(t-x)\xi]}{2\left(\beta+\sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2}\right)\left(2\xi^2+\beta^2+2\sqrt{\xi^4+\beta^2\xi^2}\right)}\,\mathrm{d}\xi, \tag{30}$$

where we have used the fact that the real part of $[K(\xi)-K_\infty(\xi)]$ is an even function of ξ, and the imaginary part is an odd function of ξ. Also note that if $\beta = 0$, then $N(x,t) = 0$. If $\beta \neq 0$, then the denominators in the two integrands of Eq. (30) never vanish for $\xi \in [0,\infty)$.

Throughout this paper, the notation $⨎$ is used for Hadamard's finite-part integral, and the notation $⨍$ is used for the Cauchy principal value (Kutt, 1975). Thus, using Eqs. (27)–(30) together with Eq. (19), one obtains a hypersingular integral equation,

$$\frac{G_0\mathrm{e}^{\beta x}}{\pi}⨎_c^d\left[\frac{1}{(t-x)^2}+\frac{\beta}{2(t-x)}+N(x,t)\right]w(t,0^+)\,\mathrm{d}t = p(x), \quad c<x<d \tag{31}$$

in which the highest order singularity defines the notation adopted, and $N(x,t)$ is given in Eq. (30). In order to have a unique solution of Eq. (31), we must impose the crack-tip conditions on $w(t,0^+)$ (Martin, 1991; Fannjiang et al., 2000):

$$w(c,0^+) = w(d,0^+) = 0. \tag{32}$$

In case of homogeneous materials, $\beta = 0$, Eq. (31) becomes

$$\frac{G_0}{\pi}⨎_c^d\frac{w(t,0^+)}{(t-x)^2}\,\mathrm{d}t = p(x), \quad c<x<d. \tag{33}$$

Therefore, one may consider the hypersingular integral equation (31) as a perturbation of Eq. (33). Also note that the expressions for $\sigma_{yz}(x,0)$, Eqs. (19) and (31), are valid for $c<x<d$ as well as for x is outside of $[c,d]$. That is,

$$\sigma_{yz}(x,0) = \frac{G_0\mathrm{e}^{\beta x}}{\pi}\int_c^d\left[\frac{1}{(t-x)^2}+\frac{\beta}{2(t-x)}+N(x,t)\right]w(t,0^+)\,\mathrm{d}t, \quad x<c \;\text{ or }\; x>d. \tag{34}$$

Note that the integrals in Eq. (34) above are ordinary integrals, not evaluated as Hadamard's finite-part integral or the Cauchy principal value integral.

6. Displacement versus slope formulations

Erdogan (1985) has studied the mode III crack problem in order to investigate the singular nature of the crack-tip stress field when the shear modulus is not smooth (continuous but not differentiable). Erdogan uses the slope function, i.e.,

$$\phi(x) = \frac{\partial}{\partial x} w(x, 0) \tag{35}$$

as the density function in formulating the governing integral equation. The resultant Cauchy singular integral equation is then solved together with the single-valuedness condition (see Eqs. (15), (20), (21), and (22) in Erdogan (1985))

$$\int_c^d \phi(x)\,\mathrm{d}x = 0. \tag{36}$$

Instead of the slope formulation, the displacement formulation is employed in this paper. Kaya (1984) has pointed out three advantages for choosing displacement over slope as the density function:

- More direct, without an extra step of integration to recover the displacement.
- The displacement function $w(x, 0)$ is bounded everywhere, but in classical linear elastic fracture mechanics (LEFM) the slope function is unbounded at the crack tips.
- Displacement would be a more natural candidate if a three-dimensional problem is considered.

Here, we point out another advantage by choosing displacement as the density function:
- *Alternative asymptotics.*

This point is important especially for the method of integral equation because the accuracy of the method relies on the exact cancellation of singularity (i.e., finite-part integrals), and a key step for achieving such cancellation is the asymptotic analysis of the kernel.

The demonstration of *alternative*, and in the present case, *simpler asymptotics* can be seen if one recalls the derivation regarding the decomposition of the kernel $K(\xi)$ described in Eqs. (24)–(26). In terms of the notation adopted in this paper, we have

$$\tilde{K}(\xi) = \lim_{y\to 0^+} \frac{\lambda(\xi)}{-\mathrm{i}\xi} \mathrm{e}^{\lambda(\xi)y} = \frac{\lambda(\xi)}{-\mathrm{i}\xi}, \quad \text{if slope formulation is used;} \tag{37}$$

$$K(\xi) = \lim_{y\to 0^+} \lambda(\xi)\mathrm{e}^{\lambda(\xi)y} = \lambda(\xi), \quad \text{if displacement formulation is taken.} \tag{38}$$

The decomposition of $\tilde{K}(\xi)$ in Eq. (37) is less straightforward. Because of the term $(-\mathrm{i}\xi)$ in the denominator in Eq. (37), one needs to consider the asymptotics of $\xi \to 0$ as well as $\xi \to \infty$. On the other hand, the decomposition of $K(\xi)$ in Eq. (38) can be achieved by considering only the asymptotics of $\xi \to \infty$:

$$\mathscr{R}(\lambda(\xi)) = \frac{-1}{\sqrt{2}} \sqrt{\sqrt{\xi^4 + \beta\xi^2} + \xi^2} \overset{|\xi|\to\infty}{\sim} -|\xi|,$$

$$\mathrm{i}\mathscr{I}(\lambda(\xi)) = \frac{-\mathrm{i}}{\sqrt{2}} \operatorname{sgn}(\beta\xi) \sqrt{\sqrt{\xi^4 + \beta\xi^2} - \xi^2} \overset{|\xi|\to\infty}{\sim} -\frac{\mathrm{i}\beta}{2}\frac{|\xi|}{\xi}.$$

The resulting singular integral equations for the two formulations are as follows:

(1) Using slope $(\partial/\partial x)w(x,0) = \phi(x)$ as density function, one obtains

$$\frac{G(x)}{\pi}\int_c^d \left[\frac{1}{t-x} + \frac{\beta}{2}\log|t-x| + \tilde{N}(x,t)\right]\phi(t)\,\mathrm{d}t = p(x), \quad c < x < d, \tag{39}$$

where $\phi(x)$ satisfies condition (36), and $\tilde{N}(x,t)$ is a regular kernel and can be found as (Erdogan, 1985, p. 824, Eqs. (24)–(27)):

$$\tilde{N}(x,t) = \int_0^\infty \frac{-\beta^4\{\cos[(t-x)\xi] - 1\}}{2\xi\left(\beta + \sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta^2\xi^2} - \xi^2}\right)\left(2\xi^2 + \beta^2 + 2\sqrt{\xi^4+\beta^2\xi^2}\right)}\,\mathrm{d}\xi$$
$$+ \int_0^\infty \frac{\beta^2\sin[(t-x)\xi]}{\left(2\xi + \sqrt{2}\sqrt{\sqrt{\xi^4+\beta^2\xi^2}+\xi^2}\right)\left(\xi + \sqrt{\xi^2+\beta^2}\right)}\,\mathrm{d}\xi. \tag{40}$$

Three comments are made here. First, the two integrals in Eq. (40) are convergent. Second, the sine integral is exactly the same as the sine integral given by Erdogan (1985, p. 824, Eqs. (26)). Moreover, although the cosine integral is different from Erdogan's (1985, p. 824, Eqs. (25)–(29)), they are equivalent due to condition (36). Third, the derivative (with respect of x) of $\tilde{N}(x,t)$ is the regular Kernel in displacement formulation. This last remark indicates the equivalence between the slope and displacement formulations.

(2) Using displacement $w(x,0)$ as density function, one obtains

$$\frac{G(x)}{\pi}\int_c^d \left[\frac{1}{(t-x)^2} + \frac{\beta}{2(t-x)} + N(x,t)\right]w(t,0^+)\,\mathrm{d}t = p(x), \quad c < x < d, \tag{41}$$

where N is given by (30) and $w(t,0^+)$ satisfies crack-tip conditions (32).

Eqs. (39) and (41) are equivalent integral equation formulations of the same boundary value problem. It is interesting to observe the similarities and difference between the displacement $N(x,t)$ and the slope $\tilde{N}(x,t)$ kernels (Eqs. (30) and (40), respectively).

7. Numerical approximation

By introducing

$$x = r(d-c)/2 + (d+c)/2, \qquad t = s(d-c)/2 + (d+c)/2, \tag{42}$$

and defining

$$\mathscr{D}(s) \equiv \frac{2}{d-c}w(t,0^+), \tag{43}$$

one may normalize Eq. (31) into

$$\frac{G_0\,\mathrm{e}^{\beta[(d-c)r/2+(d+c)/2]}}{\pi}\int_{-1}^{1}\left[\frac{1}{(s-r)^2} + \frac{\beta(d-c)}{4(s-r)} + \mathscr{N}(r,s)\right]\mathscr{D}(s)\,\mathrm{d}s = P(r), \quad |r| < 1, \tag{44}$$

where

$$\mathscr{N}(r,s) = \left(\frac{d-c}{2}\right)^2 N\left(\frac{d-c}{2}r + \frac{d+c}{2}, \frac{d-c}{2}s + \frac{d+c}{2}\right), \tag{45}$$

$$P(r) = p\left(\frac{d-c}{2}r + \frac{d+c}{2}\right). \tag{46}$$

Note that the dimensionless quantity $\beta[(d-c)/2]$ is used as the reference parameter in the numerical approximation.

According to function-theoretic method (Muskhelishvili, 1953; Erdogan et al., 1973; Erdogan, 1978; Kaya, 1984; Kaya and Erdogan, 1987), and also by using Mellin transform (Martin, 1991), the crack-tip behavior of the solution to $\mathscr{D}(s)$ in Eq. (44) has been found to be $\mathrm{O}(\sqrt{1-s^2})$ so that the $\mathscr{D}(s)$ can be written as

$$\mathscr{D}(s) = R(s)\sqrt{1-s^2}\ , \tag{47}$$

with $R(s)$ bounded for $s \in [-1, 1]$. *Usually $R(s)$ is chosen to be an expansion of Chebyshev polynomials of the second kind, $U_n(s)$, since they are a orthonormal basis with respect to the weight function $\sqrt{1-s^2}$. In this paper, Eq.* (44) *is solved numerically by the collocation method, and the orthonormal property of U_n is not essential. To make this point clear, in the next two subsections, Chebyshev polynomials expansions of both the first kind (T_n) and the second kind (U_n) are used, and the corresponding numerical results are compared.*

7.1. T_n expansion

If the function $R(s)$ in Eq. (47) is expanded by Chebyshev polynomials of the first kind, $T_n(s)$, then

$$R(s) = \sum_{n=0}^{\infty} a_n T_n(s), \tag{48}$$

and the density function $\mathscr{D}(s)$ is represented by

$$\mathscr{D}(s) = \sqrt{1-s^2}\sum_{n=0}^{\infty} a_n T_n(s), \tag{49}$$

so that the integral equation (44) becomes

$$G_0\,\mathrm{e}^{\beta[(d-c)r/2+(d+c)/2]}\sum_{n=0}^{\infty}\left\{\frac{a_n}{\pi}\,\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{T_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s + \frac{a_n}{2\pi}\left[\beta\left(\frac{d-c}{2}\right)\right]\,-\!\!\!\!\!\!\int_{-1}^{1}\frac{T_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s \right.$$
$$\left. + \frac{a_n}{2\pi}\int_{-1}^{1} T_n(s)\sqrt{1-s^2}\,\mathscr{N}(r,s)\,\mathrm{d}s\right\} = P(r), \quad -1 < r < 1. \tag{50}$$

Upon evaluation of the hypersingular and the Cauchy singular integrals (Chan et al., 2000) in the integral equation (50) above, i.e.,

$$\frac{1}{\pi}\,\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{T_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s = \begin{cases} -1, & n=0, \\ -2r, & n=1, \\ \frac{1}{2}[(n-1)U_{n-2}(r) - (n+1)U_n(r)], & n \geqslant 2, \end{cases} \tag{51}$$

$$\frac{1}{\pi}\,-\!\!\!\!\!\!\int_{-1}^{1}\frac{T_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s = \begin{cases} -r, & n=0, \\ -\frac{1}{2}(2r^2-1), & n=1, \\ \frac{1}{2}[T_{n-1}(r) - T_{n+1}(r)], & n \geqslant 2, \end{cases} \tag{52}$$

one may reach the following form for Eq. (50):

$$G_0\,\mathrm{e}^{\beta[(d-c)r/2+(d+c)/2]}\sum_{n=0}^{\infty}\left\{\frac{a_n}{2}[(n-1)U_{n-2}(r)-(n+1)U_n(r)]+\frac{a_n}{4}\left[\beta\left(\frac{d-c}{2}\right)\right][T_{n-1}(r)-T_{n+1}(r)]\right.$$
$$\left.+\frac{a_n}{2\pi}\int_{-1}^{1}T_n(s)\sqrt{1-s^2}\mathcal{N}(r,s)\,\mathrm{d}s\right\}=P(r),\quad -1<r<1, \tag{53}$$

which is used in the numerical discretization.

7.2. U_n expansion

If the function $R(s)$ in Eq. (47) is expanded by the Chebyshev polynomials of the second kind, $U_n(s)$, then

$$R(s)=\sum_{n=0}^{\infty}b_nU_n(s), \tag{54}$$

and the density function $\mathscr{D}(s)$ is represented by

$$\mathscr{D}(s)=\sqrt{1-s^2}\sum_{n=0}^{\infty}b_nU_n(s), \tag{55}$$

so that Eq. (44) becomes

$$G_0\,\mathrm{e}^{\beta[(d-c)r/2+(d+c)/2]}\sum_{n=0}^{\infty}\left\{\frac{b_n}{\pi}\oint_{-1}^{1}\frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s+\frac{b_n}{2\pi}\left[\beta\left(\frac{d-c}{2}\right)\right]\int_{-1}^{1}\frac{U_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s\right.$$
$$\left.+\frac{b_n}{2\pi}\int_{-1}^{1}U_n(s)\sqrt{1-s^2}\mathcal{N}(r,s)\,\mathrm{d}s\right\}=P(r),\quad -1<r<1. \tag{56}$$

After evaluating the hypersingular and the Cauchy singular integrals (Kaya, 1984; Kaya and Erdogan, 1987; Chan et al., 2000) in the integral equation (56) above, i.e.,

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2}\,\mathrm{d}s=-(n+1)U_n(r),\quad n\geqslant 0, \tag{57}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{U_n(s)\sqrt{1-s^2}}{s-r}\,\mathrm{d}s=-T_{n+1}(r),\quad n\geqslant 0, \tag{58}$$

one reaches

$$G_0\,\mathrm{e}^{\beta[(d-c)r/2+(d+c)/2]}\sum_{n=0}^{\infty}\left\{b_n[-(n+1)U_n(r)]+\frac{a_n}{2}\left[\beta\left(\frac{d-c}{2}\right)\right][-T_{n+1}(r)]\right.$$
$$\left.+\frac{b_n}{2\pi}\int_{-1}^{1}U_n(s)\sqrt{1-s^2}\mathcal{N}(r,s)\,\mathrm{d}s\right\}=P(r),\quad -1<r<1, \tag{59}$$

which is used in the numerical discretization.

7.3. Computational aspects

Whenever possible, the symbolical and numerical tools of the computer algebra software MAPLE (Heck, 1996) have been used for analytical manipulations. A simple numerical algorithm for solving the hypersingular integral equation (44) has been developed in MATLAB, where both T_n and U_n expressions were implemented. All the numerical results in this paper were obtained with our MATLAB code.

8. Stress intensity factors

The SIFs at the right and left crack tips are defined by

$$K_{\mathrm{III}}(d) = \lim_{x \to d^+} \sqrt{2\pi(x-d)}\sigma_{yz}(x,0), \tag{60}$$

$$K_{\mathrm{III}}(c) = \lim_{x \to c^-} \sqrt{2\pi(c-x)}\sigma_{yz}(x,0), \tag{61}$$

respectively (Fig. 1). It is worth pointing out that the limit in Eqs. (60) and (61) is taken from the outer region to the crack surfaces, and $\sigma_{yz}(x,0)$ can be evaluated by Eq. (34). The calculation of SIFs, e.g., $K_{\mathrm{III}}(d)$, is shown below. Along with the process of the derivation, some integral formulas are needed and are provided in Appendix C. The actual derivation of the integral formulas can be found in Chan et al. (2000).

By using Eqs. (60) and (34), one obtains

$$\begin{aligned} K_{\mathrm{III}}(d) &= \lim_{x \to d^+} \sqrt{2\pi(x-d)}\frac{G_0\mathrm{e}^{\beta x}}{\pi}\int_c^d \frac{w(t,0^+)}{(t-x)^2}\mathrm{d}t \\ &= \lim_{r \to 1^+} \sqrt{\pi\left(\frac{d-c}{2}\right)}\sqrt{2(r-1)}G_0 e^{\beta(d-c)r/2}\mathrm{e}^{\beta(d+c)/2}\frac{1}{\pi}\int_{-1}^{1}\frac{\mathscr{D}(s)}{(s-r)^2}\mathrm{d}s. \end{aligned} \tag{62}$$

The T_n expansion leads to

$$\begin{aligned} K_{\mathrm{III}}(d) &= \sqrt{\pi\left(\frac{d-c}{2}\right)}G_0\, e^{\beta(d-c)r/2}\mathrm{e}^{\beta(d+c)/2}\sum_0^N a_n \\ &= \sqrt{\pi\left(\frac{d-c}{2}\right)}G_0\mathrm{e}^{\beta d}\sum_0^N a_n, \end{aligned} \tag{63}$$

and the U_n expansion leads to

$$\begin{aligned} K_{\mathrm{III}}(d) &= \sqrt{\pi\left(\frac{d-c}{2}\right)}G_0\, e^{\beta(d-c)r/2}\mathrm{e}^{\beta(d+c)/2}\sum_0^N (n+1)b_n \\ &= \sqrt{\pi\left(\frac{d-c}{2}\right)}G_0\mathrm{e}^{\beta d}\sum_0^N (n+1)b_n. \end{aligned} \tag{64}$$

Similarly, one may show that with T_n expansion

$$K_{\mathrm{III}}(c) = \sqrt{\pi\left(\frac{d-c}{2}\right)} G_0\, e^{-\beta(d-c)/2} e^{\beta(d+c)/2} \sum_0^N (-1)^n a_n$$
$$= \sqrt{\pi\left(\frac{d-c}{2}\right)} G_0 e^{\beta c} \sum_0^N (-1)^n a_n, \tag{65}$$

and with U_n expansion

$$K_{\mathrm{III}}(c) = \sqrt{\pi\left(\frac{d-c}{2}\right)} G_0\, e^{-\beta(d-c)/2} e^{\beta(d+c)/2} \sum_0^N (-1)^n (n+1) b_n$$
$$= \sqrt{\pi\left(\frac{d-c}{2}\right)} G_0 e^{\beta c} \sum_0^N (-1)^n (n+1) b_n. \tag{66}$$

9. Numerical results

All the numerical results reported in this paper have been obtained by using the displacement formulation. This formulation is explained in Sections 4, 5, 7, 8 and in Appendices A–C. Fig. 2 shows the normalized SIFs for a crack in an infinite plane with shear modulus $G = G(x) = G_0 e^{\beta x}$ subjected to uniform shear traction $\sigma_{yz}(x, 0) = -p_0$. The results obtained are consistent with those of Erdogan (1985, p. 826, Fig. 2). Note that the SIFs at the tip of the stiffer side are higher than at the tip of the softer side. This result can be explained by considering the crack surface displacements (Figs. 3 and 4). For an homogeneous plane, the SIF is independent of G, i.e., $K_{\mathrm{III}}(c) = K_{\mathrm{III}}(d) = p_0\sqrt{\pi(d-c)/2}$. Nevertheless, for a nonhomogeneous plane the crack surface displacement is inversely proportional to the material parameter G (Erdogan, 1985).

Figs. 3 and 4 show numerical results for displacement profiles considering a crack with uniformly applied shear traction $\sigma_{yz}(x, 0) = -p_0$, $(c < x < d)$, and various values of the material parameter β. In Fig. 3, the cracks are tilted to the right because $\beta < 0$, and the case $\beta = 0$ corresponds to the crack surface displacement in an infinite homogeneous plane. In Fig. 4, the cracks are tilted to the left because $\beta > 0$. Figs. 3 and 4 reveal the influence of the material parameter β in the range $[-2, 2]$ on the crack profile. It can be observed that as $\beta \to 0$, the displacement profiles converge to the classical LEFM result. This numerical evidence shows that this antiplane shear problem can be considered as a perturbation of the classical antiplane shear problem for homogeneous material (Section 2).

The displacement profile in Fig. 3 with the parameter $\beta = -2$ can be compared with the result from the slope formulation (Erdogan, 1985, p. 826, Fig. 3). As expected, both results show the same magnitude and similar shape of the crack surface displacement. This confirms the agreement between slope (Erdogan, 1985) and displacement (present approach) formulations.

Table 1 presents SIFs at both tips of the crack. Note that, from a numerical point of view, essentially the same results are obtained either by the U_n or T_n representations. In Table 1, 10 decimal digits are used for the SIFs just to allow verification of this statement, otherwise, less digits should be used in reporting these results because of limitation of numerical accuracy. Such observation is made in the sense that the accuracy of numerical results should be consistent with the accuracy level provided by the approximation method being utilized.

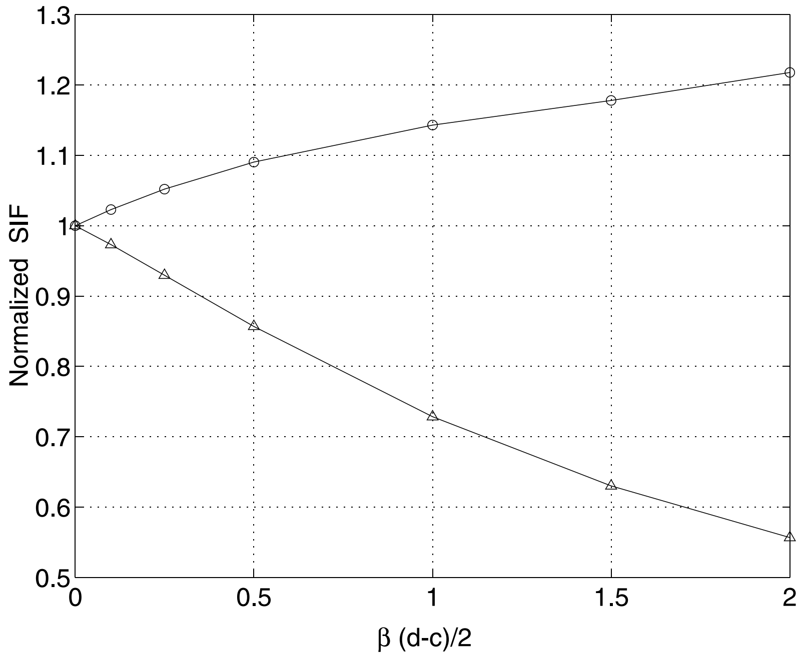

Fig. 2. Normalized stress intensity factors for an infinite nonhomogeneous plane subjected to uniform crack surface traction $\sigma_{yz}(x,0) = -p_0$. The shear modulus is $G(x) = G_0 e^{-\beta x}$. The symbol (○) stands for $K_{\mathrm{III}}(c)/(p_0\sqrt{\pi(d-c)/2})$, and the symbol (△) stands for $K_{\mathrm{III}}(d)/(p_0\sqrt{\pi(d-c)/2})$.

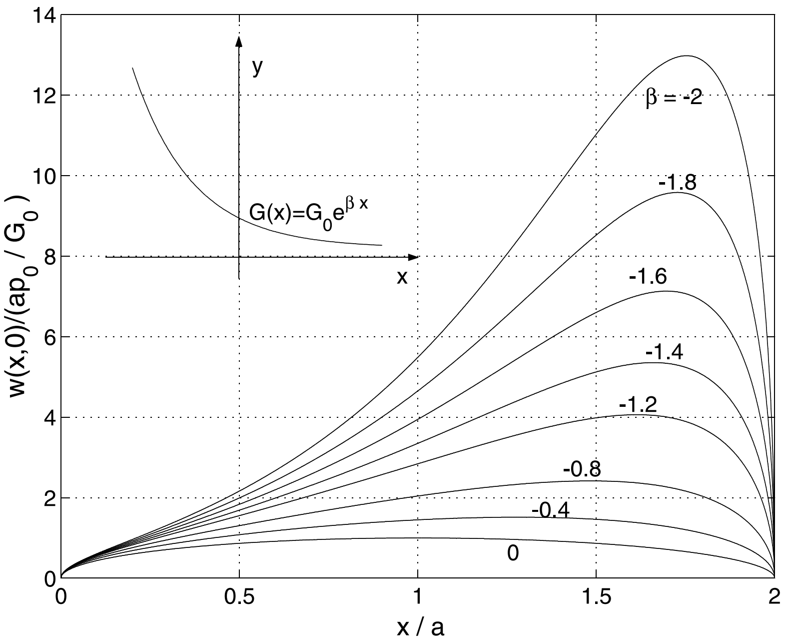

Fig. 3. Crack surface displacement in an infinite nonhomogeneous plane under uniform crack surface shear loading $\sigma_{yz}(x,0) = -p_0$ and shear modulus $G(x) = G_0 e^{\beta x}$. Here, $a = (d-c)/2$ denotes the half crack length.

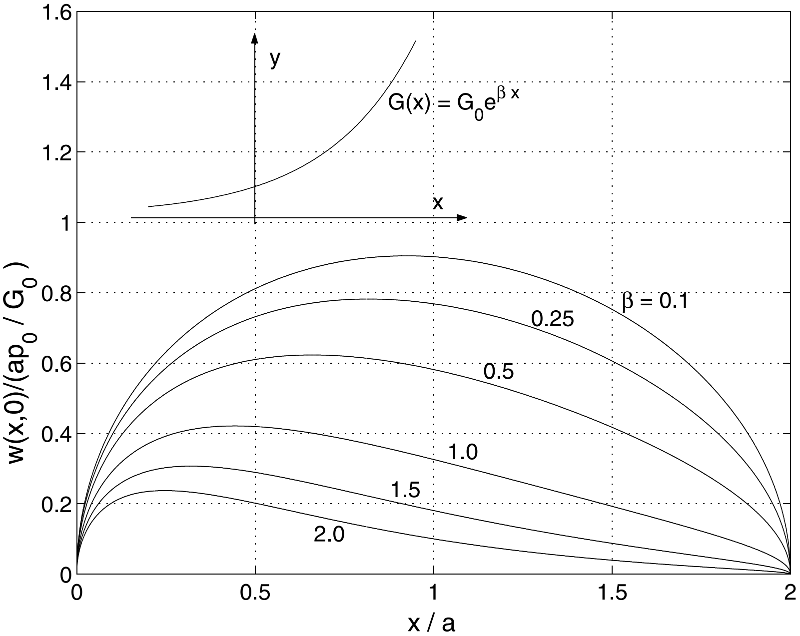

Fig. 4. Crack surface displacement in an infinite nonhomogeneous plane under uniform crack surface shear loading $\sigma_{yz}(x,0) = -p_0$ and shear modulus $G(x) = G_0 e^{\beta x}$. Here, $a = (d-c)/2$ denotes the half crack length.

Table 1
Normalized SIFs for mode III crack problem

$\beta(\frac{d-c}{2})$	U_n representation		T_n representation	
	$\frac{K_{\mathrm{III}}(c)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_{\mathrm{III}}(d)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_{\mathrm{III}}(c)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_{\mathrm{III}}(d)}{p_0\sqrt{\pi(d-c)/2}}$
−2.00	1.2177863137	0.5567159837	1.2177861733	0.5567159865
−1.50	1.1780106524	0.6300690840	1.1780106809	0.6300690822
−1.00	1.1430698167	0.7284534442	1.1430698277	0.7284534422
−0.50	1.0903639520	0.8567631803	1.0903639753	0.8567631880
−0.25	1.0518781405	0.9296196207	1.0518781461	0.9296196340
−0.10	1.0228896477	0.9731176917	1.0228896001	0.9731176549
0.00	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.10	0.9731176840	1.0228896371	0.9731176869	1.0228896411
0.25	0.9296196372	1.0518781831	0.9296196411	1.0518781724
0.50	0.8567631965	1.0903639632	0.8567631878	1.0903639710
1.00	0.7284534446	1.1430698429	0.7284534433	1.1430696546
1.50	0.6300690801	1.1780108066	0.6300690749	1.1780105468
2.00	0.5567159815	1.2177864998	0.5567159896	1.2177865939

10. Concluding remarks

A displacement based hypersingular integral equation formulation and corresponding computational implementation for the mode III crack problem in a nonhomogeneous medium with shear modulus $G(x) = G_0 \exp(\beta x)$ has been presented. We show that the displacement formulation leads to a *simpler asymptotics* in comparison with the slope-based formulation which is adopted in Erdogan (1985). Further, the displacement formulation leads to a hypersingular kernel of the type $1/(t-x)^2$ and a Cauchy kernel of the type $1/(t-x)$, while the slope formulation leads to a singular kernel of the Cauchy type and a weakly singular kernel of the logarithmic type, i.e., $\log|t-x|$. These two formulations are equivalent and lead to the same solution of the boundary value problem.

As for the solution method, it is important to remark that when the density function of the hypersingular integral equation is approximated by an expansion in Chebyshev polynomials, the finite-part integrals can be evaluated exactly (for details, see Chan et al., 2000). Hence, there is no loss of numerical accuracy due to higher-order singular integrals. The present formulation of hypersingular integral equation for crack problems in classical elasticity is motivated by crack problems in strain-gradient elasticity in which hypersingular, instead of Cauchy singular, integral equations arise even when slope is used as the density function (Paulino et al., 1999; Fannjiang et al., 2000).

Finally, this paper shows that the displacement formulation can be a sound alternative to traditional slope formulations. However, it is difficult to generalize the advantages of one formulation over the other. Thus, when an unexplored boundary value problem is solved by means of integral equations, it is worth considering alternative formulations and investigating which one is more appropriate to the actual problem at hand.

Acknowledgements

We acknowledge the support from the USA National Science Foundation (NSF) through grants CMS-9996378 (previously CMS-9713798) and DMS-9971322.

Appendix A. Derivation of the nonsingular kernel equation (26)

Recall Eq. (24). The nonsingular partition of the kernel is obtained as follows:

$$
\begin{aligned}
K(\xi)-K_\infty(\xi) &= \lambda(\xi)-\left(-|\xi|-\frac{i\beta}{2}\frac{|\xi|}{\xi}\right)\\
&= \left(|\xi|-\frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^4+\beta^2\xi^2}+\xi^2}\right)+\frac{i}{2}\frac{|\xi|}{\xi}\left(\beta-\sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta\xi^2}-\xi^2}\right)\\
&= \frac{\frac{1}{2}\xi^2-\frac{1}{2}\sqrt{\xi^4+\beta^2\xi^2}}{|\xi|+\frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^4+\beta\xi^2}+\xi^2}}+\frac{i}{2}\frac{|\xi|}{\xi}\frac{2\xi^2+\beta^2-2\sqrt{\xi^4+\beta^2\xi^2}}{\beta+\sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta\xi^2}-\xi^2}}\\
&= \frac{-\frac{1}{4}\beta^2\xi^2}{\left(|\xi|+\frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^4+\beta^2\xi^2}+\xi^2}\right)\left(\frac{1}{2}\xi^2+\frac{1}{2}\sqrt{\xi^4+\beta^2\xi^2}\right)}\\
&\quad +\frac{i}{2}\frac{|\xi|}{\xi}\frac{\beta^4}{\left(\beta+\sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta\xi^2}-\xi^2}\right)\left(2\xi^2+\beta^2+2\sqrt{\xi^4+\beta^2\xi^2}\right)}.
\end{aligned}
\tag{A.1}
$$

Dividing the first fraction in the equality numbered as Eq. (A.1) by $|\xi|\sqrt{|\xi|}$, one obtains equation (26), i.e.,

$$K(\xi)-K_\infty(\xi)=\frac{-\beta^2\sqrt{|\xi|}}{2\left(\sqrt{|\xi|}+\frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^2+\beta^2}+|\xi|}\right)\left(|\xi|+\sqrt{\xi^2+\beta^2}\right)}+\frac{\mathrm{i}\beta^4|\xi|/\xi}{2\left(\beta+\sqrt{2}\,\mathrm{sgn}(\beta)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2}\right)\left(2\xi^2+\beta^2+2\sqrt{\xi^4+\beta^2\xi^2}\right)}.$$

Note that the real part of $[K(\xi)-K_\infty(\xi)]$ is an even function of ξ, and the imaginary part is an odd function of ξ.

Appendix B. Derivation of equation (29)

It suffices to show the following two limits:

$$\int_{-\infty}^{\infty}\left[\mathrm{i}\frac{|\xi|}{\xi}\mathrm{e}^{-|\xi|y}\right]\mathrm{e}^{i(t-x)\xi}\,\mathrm{d}\xi \overset{y\to 0^+}{\to} \frac{-2}{t-x}, \tag{B.1}$$

$$\int_{-\infty}^{\infty}\left[|\xi|\mathrm{e}^{-|\xi|y}\right]\mathrm{e}^{i(t-x)\xi}\,\mathrm{d}\xi \overset{y\to 0^+}{\to} \frac{-2}{(t-x)^2}. \tag{B.2}$$

Because the derivations of both limits are very similar and the derivation of limit (B.1) is easier than Eq. (B.2), it is sufficient only to show the detail derivation of Eq. (B.2):

$$\begin{aligned}\lim_{y\to 0^+}\int_{-\infty}^{\infty}\left[|\xi|\mathrm{e}^{-|\xi|y}\right]\mathrm{e}^{\mathrm{i}(t-x)\xi}\,\mathrm{d}\xi &= \lim_{y\to 0^+}\int_{-\infty}^{\infty}\left[|\xi|\mathrm{e}^{-|\xi|y}\right]\{\cos[\xi(t-x)]+\mathrm{i}\,\sin[\xi(t-x)]\}\,\mathrm{d}\xi\\ &= \lim_{y\to 0^+} 2\int_0^{\infty}\mathrm{e}^{-\xi y}\xi\cos[\xi(t-x)]\mathrm{d}\xi\\ &= 2\lim_{y\to 0^+}\mathscr{L}\{\xi\cos[\xi(t-x)];\ \xi\to y\}\\ &= 2\lim_{y\to 0^+}(\text{Laplace transform of } \{\xi\cos[\xi(t-x)]\},\\ &\qquad \text{from variable } \xi \text{ to variable } y)\\ &= -2\lim_{y\to 0^+}\frac{(t-x)^2-y^2}{\left[(t-x)^2+y^2\right]^2}\\ &= \frac{-2}{(t-x)^2}.\end{aligned}$$

Appendix C. Useful integral formulas for deriving stress intensity factors

In this part of the appendix, the value of r is restricted to be $|r|>1$ for all the formulas given below. Therefore, each integral is not singular any more, and can easily be verified by numerical means (Chan et al., 2000).

$$\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}\mathrm{d}s = -\pi\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geqslant 0, \tag{C.1}$$

$$\int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r}\mathrm{d}s = \begin{cases} -\pi\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right), & n = 0, \\ -\frac{\pi}{2}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^2, & n = 1, \\ \pi\frac{|r|}{r}\sqrt{r^2-1}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^n, & n \geqslant 2, \end{cases} \tag{C.2}$$

$$\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2}\mathrm{d}s = -\pi(n+1)\left(1 - \frac{|r|}{\sqrt{r^2-1}}\right)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^n, \quad n \geqslant 0, \tag{C.3}$$

$$\int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^2}\mathrm{d}s = \begin{cases} -\pi\left(1 - \frac{|r|}{\sqrt{r^2-1}}\right), & n = 0, \\ -2\pi\left(1 - \frac{|r|}{\sqrt{r^2-1}}\right)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right), & n = 1, \\ \frac{\pi}{2}\left(1 - \frac{|r|}{\sqrt{r^2-1}}\right)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n-2} & \\ \quad \times\left[n - 1 - (n+1)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^2\right], & n \geqslant 2. \end{cases} \tag{C.4}$$

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