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Change of Constitutive Relations due to Interaction Between Strain-Gradient Effect and Material Gradation

For classical elasticity, the constitutive equations (Hooke's law) have the same functional form for both homogeneous and nonhomogeneous materials. However, for strain-gradient elasticity, such is not the case. This paper shows that for strain-gradient elasticity with volumetric and surface energy (Casal's continuum), extra terms appear in the constitutive equations which are associated with the interaction between the material gradation and the nonlocal effect of strain gradient. The corresponding governing partial differential equations are derived and their solutions are discussed.
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1 Introduction

In this paper we investigate the constitutive relations for strain-gradient elasticity in both homogeneous and functionally graded materials (FGMs) modeled as nonhomogeneous materials. For classical linear elasticity, the constitutive relations between the Cauchy stresses τ_{ij} and strains ε_{ij} have the same form for both homogeneous and nonhomogeneous materials. That is,

$$\tau_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}, \tag{1}$$

in which δ_{ij} is the Kronecker delta; the Lamé moduli λ and G can either be constant,

$$\lambda = \lambda_0 \quad \text{and} \quad G = G_0,$$

or they can be some functions of the material point $\mathbf{x}=(x,y,z)$,

$$\lambda \equiv \lambda(\mathbf{x}) \quad \text{and} \quad G \equiv G(\mathbf{x}).$$

While the form of the constitutive relations is the same for homogeneous or graded materials in classical elasticity, such is not the case for strain-gradient elasticity where extra terms are generated due to the interaction of strain-gradient effect and material gradation. More specifically, for homogeneous materials, the constitutive relations in strain-gradient elasticity are [1,2]

$$\tau_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} + 2G\ell' \nu_k \partial_k \varepsilon_{ij}, \tag{2}$$

where ℓ' is a material characteristic length associated with surface energy gradient, $\partial_k = \partial/\partial x_k$ is a differential operator, and ν_k, $\nu_k\nu_k = 1$, $\partial_k \nu_k = 0$, is a director field. For nonhomogeneous materials, one can NOT simply replace the Lamé moduli λ and G in Eq. (2) by the respective functions $\lambda(\mathbf{x})$ and $G(\mathbf{x})$ anymore. The corresponding constitutive equation for nonhomogeneous materials are

$$\tau_{ij} = \lambda(\mathbf{x})\varepsilon_{kk}\delta_{ij} + 2G(\mathbf{x})\varepsilon_{ij} + \ell' \nu_k [\varepsilon_{ll}\partial_k \lambda(\mathbf{x}) + \lambda(\mathbf{x})\partial_k \varepsilon_{ll}]\delta_{ij} + 2\ell' \nu_k [\varepsilon_{ij}\partial_k G(\mathbf{x}) + G(\mathbf{x})\partial_k \varepsilon_{ij}]. \tag{3}$$

Comparing Eqs. (2) and (3), one can observe that there are some extra terms in (3), and those extra terms are essentially the sum of two types of product: the product of the material gradation function [$\lambda(\mathbf{x})$ or $G(\mathbf{x})$] and the gradient of the strains, or the product of the strains and the gradient of the gradation function. It is in this sense that the extra terms are generated by the interaction of strain-gradient effect and material gradation.

Material behavior is often described by differential equations, which are formulated according to the constitutive relations. Thus, the next concern shall be how the change of constitutive equations influences the governing partial differential equations (PDEs). For instance, in classical elasticity (the constitutive relations have the same functional form for both homogeneous and nonhomogeneous materials), the governing PDEs for nonhomogeneous materials are

$$G(\mathbf{x})\nabla^2\mathbf{u} + [\lambda(\mathbf{x}) + G(\mathbf{x})]\,\nabla\,\nabla\cdot\mathbf{u} + (\nabla\mathbf{u} + \nabla\mathbf{u}^T)\,\nabla\, G(\mathbf{x}) + (\nabla\cdot\mathbf{u})\,\nabla\,\lambda(\mathbf{x}) = 0, \tag{4}$$

where $\mathbf{u}$ is the displacement vector, and ∇, $\nabla\cdot$, and ∇^2 are the gradient, divergence, and Laplacian operators, respectively. Equation (4) can be considered as a perturbation of the familiar Navier-Cauchy equations for homogeneous materials

$$G_0\nabla^2\mathbf{u} + (\lambda_0 + G_0)\,\nabla\,\nabla\cdot\mathbf{u} = 0, \tag{5}$$

where G_0 and λ_0 are the Lamé constants. Comparing Eqs. (4) and (5), one can observe that the perturbation brings in only the lower (first) order of differential operators, while the highest (second) order of differential operators has been preserved. As one of the properties of second-order elliptic PDEs, the behavior of the solution mainly depends on the highest order of the differential operators (see [3], Chap. 6). Thus, the solution to PDEs (4) should have similar behavior as the solution to PDEs (5). What is the situation for strain-gradient elasticity? It turns out that for the case of strain-gradient theory applied to FGMs, the change of PDEs is also only pertinent to the lower order differential operators, and the solution to the governing PDEs are still dominated by the highest order differential operators. In order to tell a complete story, we need to derive the governing PDEs from the equilibrium equations, in which the Cauchy stresses τ_{ij}, the couple stresses μ_{kij}, and the total stresses σ_{ij} are all involved. Thus, we need to know all the constitutive relations between strains and each of the stress fields. In this work the derivation of constitutive relations in strain-gradient elasticity relies on the strain-energy density function W.

This paper presents a detailed derivation of the constitutive relations in strain-gradient elasticity and the corresponding govern-

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ing PDEs. The paper is organized as follows. First, the strain-energy density function W is introduced, the constitutive relations are derived from first principles, and some remarks about admissibility of W are made. Then, the governing PDEs of strain-gradient elasticity for anti-plane shear problems and plane state problems are derived. The behavior of the solutions to the corresponding PDEs are discussed. Finally, some concluding remarks are given at the end of the paper.

2 Strain-Energy Density Function

2.1 Elasticity. In classical elasticity, the strain-energy density function has the well-known form

$$W = \tfrac{1}{2}\lambda(\mathbf{x})\varepsilon_{ii}\varepsilon_{jj} + G(\mathbf{x})\varepsilon_{ij}\varepsilon_{ji}, \tag{6}$$

where $\lambda(\mathbf{x})$ and $G(\mathbf{x})$ are the material parameters which are functions of position $\mathbf{x}$, and ε is the small deformation tensor

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{7}$$

with $\mathbf{u}$ denoting the displacement vector. The Cauchy stresses are given by Eq. (1), i.e.,

$$\tau_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \lambda(\mathbf{x})\varepsilon_{kk}\delta_{ij} + 2G(\mathbf{x})\varepsilon_{ij}. \tag{8}$$

In the case of homogeneous materials, λ and G are constants (Lamé constants) and the Cauchy stresses, derived from (6), is

$$\tau_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}. \tag{9}$$

Notice that Eqs. (8) and (9) have the same functional form.

2.2 Gradient Elasticity. Casal's anisotropic grade-2 elasticity theory is used in this paper; as an analogue to the concept of the surface tension of liquid, two material constants, the volume strain-gradient term ℓ and the surface energy strain-gradient term ℓ', were introduced by Casal to characterize the internal and surface capillarity of the solid. The surface energy strain-gradient term ℓ' cannot exist alone (i.e., $\ell=0$ and $\ell' \neq 0$ is not an admissible configuration) because the strain-energy density function needs to be non-negative. The effect of the volume strain-gradient term ℓ is to shield the applied loads leading to crack stiffening, and the effect of the surface energy strain-gradient term ℓ' is to amplify the applied loads leading to crack compliance by increasing the energy release rate of the crack [4]. The ratio $\rho=\ell'/\ell$ has been investigated in detail by Fannjiang et al. [5].

The three-dimensional generalization of Casal's gradient-dependent anisotropic elasticity with volumetric and surface energy for nonhomogeneous materials leads to the following expression for the strain-energy density function:

$$\begin{aligned} W = {} & \tfrac{1}{2}\lambda(\mathbf{x})\varepsilon_{ii}\varepsilon_{jj} + G(\mathbf{x})\varepsilon_{ij}\varepsilon_{ji} + \tfrac{1}{2}\lambda(\mathbf{x})\ell^2(\partial_k\varepsilon_{ii})(\partial_k\varepsilon_{jj}) \\ & + \tfrac{1}{2}\ell'\nu_k\partial_k[\lambda(\mathbf{x})\varepsilon_{ii}\varepsilon_{jj}] + G(\mathbf{x})\ell^2(\partial_k\varepsilon_{ij})(\partial_k\varepsilon_{ji}) \\ & + \ell'\nu_k\partial_k[G(\mathbf{x})\varepsilon_{ij}\varepsilon_{ji}], \quad \ell > 0, \end{aligned} \tag{10}$$

where ℓ and ℓ' are two material characteristic lengths associated with volumetric and surface energy gradient terms, respectively. The terms associated with ℓ' have the meaning of surface energy. It is easy to see that, after integrating W over the material domain Ω and applying the divergence theorem with $\partial_k\nu_k=0$, the terms associated with ℓ' become surface integrals,[1] i.e.,

$$\begin{aligned} & \int_\Omega \left(\frac{1}{2}\ell'\nu_k\partial_k[\lambda(\mathbf{x})\varepsilon_{ii}\varepsilon_{jj}] + \ell'\nu_k\partial_k[G(\mathbf{x})\varepsilon_{ij}\varepsilon_{ji}]\right)dV \\ & = \ell'\int_{\partial\Omega}\left(\frac{1}{2}\lambda(\mathbf{x})(\varepsilon_{ii}\varepsilon_{jj})(\nu_k n_k) + G(\mathbf{x})(\varepsilon_{ij}\varepsilon_{ij})(\nu_k n_k)\right)dS, \end{aligned} \tag{11}$$

where n_k is the outward unit normal to the boundary $\partial\Omega$. By considering the particular case $\nu_k \equiv n_k$, the director field has the same direction as the outward unit normal to the boundary, the surface integral simply becomes

$$\ell'\int_{\partial\Omega}\left(\frac{1}{2}\lambda(\mathbf{x})(\varepsilon_{ii}\varepsilon_{jj}) + G(\mathbf{x})(\varepsilon_{ij}\varepsilon_{ij})\right)dS. \tag{12}$$

By definition, the Cauchy stresses τ_{ij}, the couple stresses μ_{kij}, and the total stresses σ_{ij} are

$$\begin{aligned} \tau_{ij} &= \partial W/\partial\varepsilon_{ij} \\ \mu_{kij} &= \partial W/\partial\varepsilon_{ij,k} \\ \sigma_{ij} &= \tau_{ij} - \partial_k\mu_{kij}. \end{aligned} \tag{13}$$

Using Eqs. (13) and (10), the constitutive equations for functionally graded materials are

$$\begin{aligned} \tau_{ij} = {} & \lambda(\mathbf{x})\varepsilon_{kk}\delta_{ij} + 2G(\mathbf{x})\varepsilon_{ij} + \ell'\nu_k[\varepsilon_{ll}\partial_k\lambda(\mathbf{x}) + \lambda(\mathbf{x})\partial_k\varepsilon_{ll}]\delta_{ij} \\ & + 2\ell'\nu_k[\varepsilon_{ij}\partial_k G(\mathbf{x}) + G(\mathbf{x})\partial_k\varepsilon_{ij}] \end{aligned} \tag{14}$$

$$\mu_{kij} = \ell'\nu_k\lambda(\mathbf{x})\varepsilon_{ll}\delta_{ij} + \ell^2\lambda(\mathbf{x})\partial_k\varepsilon_{ll}\delta_{ij} + 2\ell'\nu_k G(\mathbf{x})\varepsilon_{ij} + 2\ell^2 G(\mathbf{x})\partial_k\varepsilon_{ij} \tag{15}$$

$$\begin{aligned} \sigma_{ij} = {} & \lambda(\mathbf{x})(\varepsilon_{kk} - \ell^2\nabla^2\varepsilon_{kk})\delta_{ij} + 2G(\mathbf{x})(\varepsilon_{ij} - \ell^2\nabla^2\varepsilon_{ij}) \\ & - \ell^2[\partial_k\lambda(\mathbf{x})](\partial_k\varepsilon_{ll})\delta_{ij} - 2\ell^2[\partial_k G(\mathbf{x})](\partial_k\varepsilon_{ij}) \end{aligned} \tag{16}$$

2.3 Remarks. If the material is homogeneous, then the Lamé constants λ and G in Eq. (10) can be placed either before or after the differential operator $\partial_k=\partial/\partial x_k$. However, if the material is nonhomogeneous, then different positions of λ and G in Eq. (10) would lead to different strain-energy density functions. Thus, if one expresses the strain-energy density as

$$\begin{aligned} W_A \equiv {} & \tfrac{1}{2}\lambda(\mathbf{x})\varepsilon_{ii}\varepsilon_{jj} + G(\mathbf{x})\varepsilon_{ij}\varepsilon_{ji} + \tfrac{1}{2}\ell^2\partial_k[\lambda(\mathbf{x})\varepsilon_{ii}](\partial_k\varepsilon_{jj}) \\ & + \tfrac{1}{2}\ell'\nu_k\partial_k[\lambda(\mathbf{x})\varepsilon_{ii}\varepsilon_{jj}] + \ell^2\partial_k[G(\mathbf{x})\varepsilon_{ij}](\partial_k\varepsilon_{ji}) \\ & + \ell'\nu_k\partial_k[G(\mathbf{x})\varepsilon_{ij}\varepsilon_{ji}], \end{aligned} \tag{17}$$

then it is clear that by the product rule of derivative, W_A and W are different. Two other strain-energy density expressions can be obtained by placing $\lambda(\mathbf{x})$ and $G(\mathbf{x})$, the Lamé moduli associated with the surface characteristic length ℓ', in front of the differential operator ∂_k in Eqs. (10) and (17) [6]. We choose to work with W because it gives rise to an energy functional that is always positive-definite regardless of the material inhomogeneities $(\lambda(\mathbf{x}), G(\mathbf{x}))$ and the strain-gradient parameters $\ell, \ell' \geqslant 0$. When the material inhomogeneities are present and rough [i.e., the derivatives of $\lambda(\mathbf{x})$, $G(\mathbf{x})$ are sufficiently large] the other (three) energy functionals lose positive-definiteness, resulting in negative total energy of possibly arbitrary magnitudes. Thus, in this paper we restricted our consideration to the energy density W and derive the constitutive relations and the corresponding PDEs from it.

[1] To get Eq. (11), one needs to specify the director field in the interior as well, namely, it has to be divergence free. If one allows non-divergence-free director field, then it is possible to have λ and G standing out of the partial derivative in the ℓ' terms of (10) and still representing surface energy.

3 Plane State Problems

In this section we derive the governing (system of) PDEs of gradient elasticity for a plane problem in functionally graded materials from the strain-energy density function. The process is similar to the one for anti-plane shear case, however the algebra is more involved.

3.1 Constitutive Equations. From the definition of τ_{ij}, μ_{kij}, and σ_{ij} in Eq. (13), we have already obtained the (general plane) constitutive equations of gradient elasticity for FGMs in Eqs. (14)–(16). For homogeneous materials, the constitutive equations are [1,7]

$$\tau_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} + \ell' \nu_k \partial_k(\lambda \varepsilon_{ll}\delta_{ij} + 2G\varepsilon_{ij}) \tag{18}$$

$$\mu_{kij} = \lambda \ell^2 \partial_k \varepsilon_{ll}\delta_{ij} + 2G\ell' \nu_k \varepsilon_{ij} + \lambda \ell' \nu_k \varepsilon_{ll}\delta_{ij} + 2G\ell^2 \partial_k \varepsilon_{ij} \tag{19}$$

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} - \ell^2\nabla^2(\lambda \varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}). \tag{20}$$

Comparing Eqs. (14)–(16) with (18)–(20), one notices that the couple stresses μ_{kij} in (15) and (19) take the same form. However, for the total stresses σ_{ij}, there are more terms in (16) than in (20), and those extra terms will confound the form of the governing (system of) PDEs.

For two-dimensional plane problems, the components of the strain tensor are given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right), \quad \varepsilon_{xz} = \varepsilon_{yz} = \varepsilon_{zz} = 0. \tag{21}$$

The components of the stress fields for homogeneous materials are [7]

$$\begin{aligned}
&\sigma_{xz} = \sigma_{yz} = 0 \\
&\sigma_{xx} = (\lambda + 2G)\varepsilon_{xx} + \lambda \varepsilon_{yy} - (\lambda + 2G)\ell^2\nabla^2\varepsilon_{xx} - \lambda \ell^2\nabla^2\varepsilon_{yy} \\
&\sigma_{yy} = (\lambda + 2G)\varepsilon_{yy} + \lambda \varepsilon_{xx} - (\lambda + 2G)\ell^2\nabla^2\varepsilon_{yy} - \lambda \ell^2\nabla^2\varepsilon_{xx} \\
&\sigma_{xy} = \sigma_{yx} = 2G\varepsilon_{xy} - 2G\ell^2\nabla^2\varepsilon_{xy} \\
&\sigma_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) - \lambda \ell^2\nabla^2(\varepsilon_{xx} + \varepsilon_{yy}),
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
&\mu_{xxx} = (\lambda + 2G)\ell^2\partial_x\varepsilon_{xx} + \lambda \ell^2\partial_x\varepsilon_{yy} \\
&\mu_{yxx} = -(\lambda + 2G)\ell'\varepsilon_{xx} - \lambda \ell'\varepsilon_{yy} + (\lambda + 2G)\ell^2\partial_y\varepsilon_{xx} + \lambda \ell^2\partial_y\varepsilon_{yy} \\
&\mu_{xyy} = (\lambda + 2G)\ell^2\partial_x\varepsilon_{yy} + \lambda \ell^2\partial_x\varepsilon_{xx} \\
&\mu_{yyy} = -(\lambda + 2G)\ell'\varepsilon_{yy} - \lambda \ell'\varepsilon_{xx} + (\lambda + 2G)\ell^2\partial_y\varepsilon_{yy} + \lambda \ell^2\partial_y\varepsilon_{xx} \\
&\mu_{xxy} = \mu_{xyx} = 2G\ell^2\partial_x\varepsilon_{xy} \\
&\mu_{yyx} = \mu_{yxy} = -2G\ell'\varepsilon_{xy} + 2G\ell^2\partial_y\varepsilon_{xy}.
\end{aligned} \tag{23}$$

For nonhomogeneous materials, the couple stresses μ_{kij} have the same form as in (23), except that the Lamé constants functional λ and G are not constants, they are functions of (x,y) according to the gradation of the material. The total stresses σ_{ij} have more terms than in (22) and they are

$$\begin{aligned}
&\sigma_{xz} = \sigma_{yz} = 0 \\
&\sigma_{xx} = [\lambda(x,y) + 2G(x,y)](1 - \ell^2\nabla^2)\varepsilon_{xx} + \lambda(x,y)(1 - \ell^2\nabla^2)\varepsilon_{yy} \\
&\quad - \ell^2\{[\partial_x\lambda(x,y)]\partial_x(\varepsilon_{xx} + \varepsilon_{yy}) + [\partial_y\lambda(x,y)]\partial_y(\varepsilon_{xx} + \varepsilon_{yy})\} \\
&\quad - 2\ell^2\{[\partial_x G(x,y)]\partial_x\varepsilon_{xx} + [\partial_y G(x,y)]\partial_y\varepsilon_{xx}\} \\
&\sigma_{yy} = [\lambda(x,y) + 2G(x,y)](1 - \ell^2\nabla^2)\varepsilon_{yy} + \lambda(x,y)(1 - \ell^2\nabla^2)\varepsilon_{xx} \\
&\quad - \ell^2\{[\partial_x\lambda(x,y)]\partial_x(\varepsilon_{xx} + \varepsilon_{yy}) + [\partial_y\lambda(x,y)]\partial_y(\varepsilon_{xx} + \varepsilon_{yy})\} \\
&\quad - 2\ell^2\{[\partial_x G(x,y)]\partial_x\varepsilon_{yy} + [\partial_y G(x,y)]\partial_y\varepsilon_{yy}\} \\
&\sigma_{xy} = \sigma_{yx} = 2G(x,y)(\varepsilon_{xy} - \ell^2\nabla^2\varepsilon_{xy}) - 2\ell^2\{[\partial_x G(x,y)]\partial_x\varepsilon_{xy} \\
&\quad + [\partial_y G(x,y)]\partial_y\varepsilon_{xy}\} \\
&\sigma_{zz} = \lambda(x,y)[(\varepsilon_{xx} + \varepsilon_{yy}) - \ell^2\nabla^2(\varepsilon_{xx} + \varepsilon_{yy})] - \ell^2\{[\partial_x\lambda(x,y)]\partial_x(\varepsilon_{xx} \\
&\quad + \varepsilon_{yy}) + [\partial_y\lambda(x,y)]\partial_y(\varepsilon_{xx} + \varepsilon_{yy})\}
\end{aligned} \tag{24}$$

3.2 Governing System of PDEs. By imposing the equilibrium equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0, \tag{25}$$

and using Eqs. (21) and (24), one obtains the following system of PDEs:

$$\begin{aligned}
&G(x,y)\nabla^2(1 - \ell^2\nabla^2)\mathbf{u} + [\lambda(x,y) + G(x,y)]\,\nabla\,(1 - \ell^2\nabla^2)\,\nabla \cdot \mathbf{u} + [(1 \\
&\quad - \ell^2\nabla^2)(\nabla\mathbf{u} + \nabla\mathbf{u}^T)]\,\nabla\, G(x,y) + [(1 - \ell^2\nabla^2)\,\nabla \cdot \mathbf{u}]\,\nabla\, \lambda(x,y) \\
&\quad - \ell^2\Bigg[\left(\nabla\frac{\partial}{\partial x}\mathbf{u}\right)\nabla\,\frac{\partial G(x,y)}{\partial x} + \left(\nabla\frac{\partial}{\partial y}\mathbf{u}\right)\nabla\,\frac{\partial G(x,y)}{\partial y} \\
&\quad - \nabla[\nabla\lambda(x,y)\cdot\nabla\nabla\cdot\mathbf{u}]\Bigg] - \ell^2\Bigg(\frac{\partial}{\partial x}[(\nabla\nabla u)\,\nabla\, G(x,y)] \\
&\quad + \frac{\partial}{\partial y}[(\nabla\nabla v)\,\nabla\, G(x,y)] + (\nabla\nabla^2\mathbf{u})\,\nabla\, G(x,y)\Bigg) = 0,
\end{aligned} \tag{26}$$

where the boldface $\mathbf{u}$ denotes the displacement vector (u,v). Equation (26) is the most general form. In particular, if the moduli vary as a function of (x,y) and assume the exponential form

$$G \equiv G(x,y) = G_0 e^{\beta x + \gamma y}, \quad \lambda \equiv \lambda(x,y) = \frac{3-\kappa}{\kappa - 1}G(x,y), \tag{27}$$

then the system of PDEs is

$$\begin{aligned}
&\left(1 - \beta\ell^2\frac{\partial}{\partial x} - \gamma\ell^2\frac{\partial}{\partial y} - \ell^2\nabla^2\right)\Bigg((\kappa+1)\frac{\partial^2 u}{\partial x^2} + (\kappa-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x\partial y} \\
&\quad + \beta(\kappa+1)\frac{\partial u}{\partial x} + \gamma(\kappa-1)\frac{\partial u}{\partial y} + \gamma(\kappa-1)\frac{\partial v}{\partial x} + \beta(3-\kappa)\frac{\partial v}{\partial y}\Bigg) = 0,
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\left(1 - \beta\ell^2\frac{\partial}{\partial x} - \gamma\ell^2\frac{\partial}{\partial y} - \ell^2\nabla^2\right)\Bigg((\kappa-1)\frac{\partial^2 v}{\partial x^2} + (\kappa+1)\frac{\partial^2 v}{\partial y^2} + 2\frac{\partial^2 u}{\partial x\partial y} \\
&\quad + \gamma(3-\kappa)\frac{\partial u}{\partial x} + \beta(\kappa-1)\frac{\partial u}{\partial y} + \beta(\kappa-1)\frac{\partial v}{\partial x} + \gamma(\kappa+1)\frac{\partial v}{\partial y}\Bigg) = 0,
\end{aligned} \tag{29}$$

where $\kappa = 3 - 4\nu$ if plane strain is considered, $\kappa = (3-\nu)/(1+\nu)$ if it is a plane stress problem, and ν is the Poisson's ratio.

If G and λ are constants, then the homogeneous material case is recovered, and the system of PDEs (26) is reduced to

$$(1 - \ell^2\nabla^2)[G\nabla^2\mathbf{u} + (\lambda + G)\,\nabla\,\nabla\cdot\mathbf{u}] = 0, \tag{30}$$

which has been studied by Exadaktylos [7]. In the conventional classical linear elasticity (i.e., $\ell \to 0$), the system of PDEs (26) becomes (4). If G and λ take the form in (27), then (4) can be expressed as

$$\begin{aligned}
&(\kappa+1)\frac{\partial^2 u}{\partial x^2} + (\kappa-1)\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 v}{\partial x\partial y} + \beta(\kappa+1)\frac{\partial u}{\partial x} + \gamma(\kappa-1)\frac{\partial u}{\partial y} \\
&\quad + \gamma(\kappa-1)\frac{\partial v}{\partial x} + \beta(3-\kappa)\frac{\partial v}{\partial y} = 0
\end{aligned} \tag{31}$$

$$(\kappa-1)\frac{\partial^2 v}{\partial x^2}+(\kappa+1)\frac{\partial^2 v}{\partial y^2}+2\frac{\partial^2 u}{\partial x\partial y}+\gamma(3-\kappa)\frac{\partial u}{\partial x}+\beta(\kappa-1)\frac{\partial u}{\partial y}$$

$$+\beta(\kappa-1)\frac{\partial v}{\partial x}+\gamma(\kappa+1)\frac{\partial v}{\partial y}=0. \quad (32)$$

This system (31) and (32) has been studied by Konda and Erdogan [8]; for the homogeneous materials, they can be further simplified to Navier-Cauchy equations (5) for the elastic medium.

4 Anti-Plane Shear

In this section we derive the governing PDE of gradient elasticity for an anti-plane shear problem in functionally graded materials. It is worth mentioning that this type of problem has attracted the attention of several researchers, such as Vardoulakis et al. [2], Fannjiang et al. [5], Georgiadis [9], and Zhang et al. [10].

4.1 Constitutive Equations. In three-dimensional space, the displacement components are defined as

$$u_x \equiv u, \quad u_y \equiv v, \quad u_z \equiv w. \quad (33)$$

As in Eq. (7), strains are defined by

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \quad (34)$$

where both the indices i and j run through x, y, and z. The strain-energy density function (for anti-plane shear) is

$$W=\tfrac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj}+G\varepsilon_{ij}\varepsilon_{ji}+\ell^2G(\partial_k\varepsilon_{ij})(\partial_k\varepsilon_{ji})+\ell'\nu_k\partial_k(G\varepsilon_{ij}\varepsilon_{ji}). \quad (35)$$

We define the Cauchy stresses τ_{ij}, the couple stresses μ_{kij}, and the total stresses σ_{ij} according to equations in (13). Thus, the constitutive equations of gradient elasticity in anti-plane problems for homogeneous materials can be directly derived as [1,2]

$$\tau_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2G\varepsilon_{ij}+2G\ell'\nu_k\partial_k\varepsilon_{ij} \quad (36)$$

$$\mu_{kij}=2G\ell'\nu_k\varepsilon_{ij}+2G\ell^2\partial_k\varepsilon_{ij} \quad (37)$$

$$\sigma_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2G(\varepsilon_{ij}-\ell^2\nabla^2\varepsilon_{ij}). \quad (38)$$

For functionally graded materials the corresponding constitutive equations are

$$\tau_{ij}=\lambda(\mathbf{x})\varepsilon_{kk}\delta_{ij}+2G(\mathbf{x})\varepsilon_{ij}+2\ell'\nu_k[\varepsilon_{ij}\partial_kG(\mathbf{x})+G(\mathbf{x})\partial_k\varepsilon_{ij}] \quad (39)$$

$$\mu_{kij}=2\ell'\nu_kG(\mathbf{x})\varepsilon_{ij}+2\ell^2G(\mathbf{x})\partial_k\varepsilon_{ij} \quad (40)$$

$$\sigma_{ij}=\lambda(\mathbf{x})\varepsilon_{kk}\delta_{ij}+2G(\mathbf{x})(\varepsilon_{ij}-\ell^2\nabla^2\varepsilon_{ij})-2\ell^2[\partial_kG(\mathbf{x})](\partial_k\varepsilon_{ij}). \quad (41)$$

It is worth pointing out that in each of (39) and (41), there is an extra term with respect to (36) and (38), respectively. The extra terms will disappear if there is no material gradation. Thus, for homogeneous materials, Eqs. (39)–(41) will become the same as (36)–(38).

According to the relations in (36)–(38), each component of the stress fields for homogeneous materials can be written as [2]

$$\sigma_{xx}=\sigma_{yy}=\sigma_{zz}=0, \quad \sigma_{xy}=0$$

$$\sigma_{xz}=2G(\varepsilon_{xz}-\ell^2\nabla^2\varepsilon_{xz})\neq 0, \quad \sigma_{yz}=2G(\varepsilon_{yz}-\ell^2\nabla^2\varepsilon_{yz})\neq 0$$

$$\mu_{xxz}=2G\ell^2\partial_x\varepsilon_{xz}, \quad \mu_{xyz}=2G\ell^2\partial_x\varepsilon_{yz}$$

$$\mu_{yxz}=2G(\ell^2\partial_y\varepsilon_{xz}-\ell'\varepsilon_{xz}), \quad \mu_{yyz}=2G(\ell^2\partial_y\varepsilon_{yz}-\ell'\varepsilon_{yz}). \quad (42)$$

For FGMs, from the relations in (39)–(41), each component of the stress fields is found to be

$$\sigma_{xx}=\sigma_{yy}=\sigma_{zz}=0, \quad \sigma_{xy}=0$$

$$\sigma_{xz}=2G(x,y)(\varepsilon_{xz}-\ell^2\nabla^2\varepsilon_{xz})-2\ell^2\{[\partial_xG(x,y)](\partial_x\varepsilon_{xz})+[\partial_yG(x,y)](\partial_y\varepsilon_{xz})\}\neq 0$$

$$\sigma_{yz}=2G(x,y)(\varepsilon_{yz}-\ell^2\nabla^2\varepsilon_{yz})-2\ell^2\{[\partial_xG(x,y)](\partial_x\varepsilon_{yz})+[\partial_yG(x,y)](\partial_y\varepsilon_{yz})\}\neq 0 \quad (43)$$

$$\mu_{xxz}=2G(x,y)\ell^2\partial_x\varepsilon_{xz}, \quad \mu_{xyz}=2G(x,y)\ell^2\partial_x\varepsilon_{yz}$$

$$\mu_{yxz}=2G(x,y)(\ell^2\partial_y\varepsilon_{xz}-\ell'\varepsilon_{xz}), \quad \mu_{yyz}=2G(x,y)(\ell^2\partial_y\varepsilon_{yz}-\ell'\varepsilon_{yz}).$$

Again, comparing Eqs. (42) and (43), one notices that there are extra terms in the total stresses σ_{ij} of (43) due to the interaction of material gradation and the nonlocal effect of strain gradient. As the equilibrium equation only involves σ_{ij} [see Eq. (46)], the extra terms will complicate the governing PDE(s) a bit more. The couple stresses μ_{kij} in (42) and (43) assume the same form, except that G in (43) is not a constant, but rather a function reflecting the gradation of the material.

4.2 Governing PDE. For an anti-plane problem, the following relations hold:

$$u=0, \quad v=0, \quad w=w(x,y), \quad (44)$$

where u, v, and w denote the displacement components along the axes x, y, and z, respectively. The nontrivial strains are

$$\varepsilon_{xz}=\frac{1}{2}\frac{\partial w}{\partial x}, \quad \varepsilon_{yz}=\frac{1}{2}\frac{\partial w}{\partial y}. \quad (45)$$

By imposing the equilibrium equation

$$\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}=0 \quad (46)$$

with the expressions σ_{xz} and σ_{yz} in (43), one obtains the following PDE:

$$\nabla G(x,y)\cdot(1-\ell^2\nabla^2)\nabla w+G(x,y)(1-\ell^2\nabla^2)\nabla^2 w-\ell^2\left(\nabla\frac{\partial G(x,y)}{\partial x}\cdot\nabla\frac{\partial w}{\partial x}+\nabla\frac{\partial G(x,y)}{\partial y}\cdot\nabla\frac{\partial w}{\partial y}+\nabla G(x,y)\cdot\nabla\nabla^2 w\right)=0. \quad (47)$$

If G is an exponential function of both x and y,

$$G\equiv G(x,y)=G_0e^{\beta x+\gamma y}, \quad (48)$$

then the governing PDE is

$$\left(1-\beta\ell^2\frac{\partial}{\partial x}-\gamma\ell^2\frac{\partial}{\partial y}-\ell^2\nabla^2\right)\left(\nabla^2+\beta\frac{\partial}{\partial x}+\gamma\frac{\partial}{\partial y}\right)w=0. \quad (49)$$

In Table 1 we list the governing PDEs in anti-plane shear problems that correspond to different combinations of parameter ℓ and various material gradation of the shear modulus G.

5 Further Remarks

The conventional continuum mechanics theories have been used adequately when the length scale of the deformation field is much larger than the underlying micro-structure length scale of the material. As the two length scales become comparable, the material behavior at one point tends to be influenced more significantly by the neighboring material points. The criterion for adopting the strain gradient theory should depend on the experimental

Table 1 Governing PDEs in antiplane shear problems

Cases	Governing PDE	References
$\ell=0,\ G\equiv\text{const}$	Laplace equation: $\nabla^2 w=0$	Standard textbooks
$\ell=0,\ G\equiv G(y)=G_0 e^{\gamma y}$	Perturbed Laplace equation: $\left(\nabla^2+\gamma\frac{\partial}{\partial y}\right)w=0$	Erdogan and Ozturk [11]
$\ell=0,\ G\equiv G(x)=G_0 e^{\beta x}$	Perturbed Laplace equation: $\left(\nabla^2+\beta\frac{\partial}{\partial x}\right)w=0$	Chan et al. [12] Erdogan [13]
$\ell\neq 0,\ G\equiv\text{const}$	Helmholtz-Laplace equation: $(1-\ell^2\nabla^2)\nabla^2 w=0$	Vardoulakis et al. [2] Fannjiang et al. [5] Zhang et al. [10]
$\ell\neq 0,\ G\equiv G(y)=G_0 e^{\gamma y}$	$\left(1-\gamma\ell^2\frac{\partial}{\partial y}-\ell^2\nabla^2\right)\left(\nabla^2+\gamma\frac{\partial}{\partial y}\right)w=0$	Paulino et al. [14]
$\ell\neq 0,\ G\equiv G(x)=G_0 e^{\beta x}$	$\left(1-\beta\ell^2\frac{\partial}{\partial x}-\ell^2\nabla^2\right)\left(\nabla^2+\beta\frac{\partial}{\partial x}\right)w=0$	Chan et al. [15]
$\ell\neq 0$, general $G\equiv G(x,y)$	Eq. (47)	Not available

data, and there are many experiments indicating conventional continuum mechanics cannot lead to a satisfactory prediction of the material behavior as the two length scales mentioned above are comparable to each other. Experimental techniques related to strain gradient theory include micro-torsion [16], micro-bending [17], and micro-indentation [18], which can be associated to the parameter ℓ. However, the authors are not aware of experiments associated directly to ℓ', which indicates an area for further research.

The inhomogeneity of materials can be caused by many mechanisms in different length scales, such as the size and distribution of inclusions, the grain size of crystals, and the size of constituent atoms and molecules. Thus a constant ℓ' cannot describe these different length scales. Ideally $\ell'\equiv\ell'(\mathbf{x})$; however, here we consider the gradient parameters ℓ' and ℓ as constants.

6 Conclusion

In the conventional classical linear elasticity, one may derive the governing PDE(s) for nonhomogeneous materials by directly replacing the Lamé constants with the material gradation functions at the level of the constitutive equations. We have shown that this is not the case for strain gradient elasticity because extra terms may arise. These extra terms come from the interaction between the material gradation and the nonlocal effect of the strain gradient. Thus, the constitutive equations for nonhomogeneous materials are different from the ones for homogeneous materials under the consideration of strain gradient elasticity theory (Casal's continuum). The governing PDEs for nonhomogeneous materials are derived by means of the strain energy density function and the corresponding definitions of the stress fields (which have been presented in this paper).

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